PhD subject – 2026
ALARICE – ANR-24-CE48-7504
General complexity bounds for finite dynamical systems

Simulations as reductions

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Spirit Rice proved in 1953 one of the most meaningful theorem in computer science:

Theorem ([1])**.** *Any non-trivial question on the behavior of programs is undecidable.*

What is striking about this statement is its generality: it holds for *any* non-trivial question. We aim at obtaining analogous results in the decidable world *i.e.*, general (lower and upper) bounds of computational complexity. Why this shift? Because the objects we will consider are finite dynamical systems, where virtually any question is decidable via a naïve exhaustive search.

Automata networks Automata networks (ANs) offer a very expressive framework: any digraph is the dynamics of some AN, hence any finite (discrete) dynamical system can be modeled as an AN. Formally, an AN is composed of n automata, each holding a state among a finite alphabet A_i for $i \in [n] = \{1, \ldots, n\}$, and a *local function* $f_i : X \to A_i$ giving its next state provided the current global state of the system within $X = \Pi_{i\in[n]} A_i$. We have a dynamical system (X, f), where $f : X \to X$ is defined as $\forall x \in X : f(x) = (f_1(x), \ldots, f_n(x))$ *i.e.*, all automata update their state in parallel (*fully synchronous* mode) at each step. Other update policies may be observed (including *non-deterministic* update modes). The graph of f on vertex set X is its *dynamics*, denoted $\mathscr{G}_f$ (it has out-degree one in deterministic settings). When $A_1 = \ldots = A_n = \{0, 1\}$, we have *Boolean automata networks* (BANs), on which we will stick for simplicity in order to introduce a central object of AN theory: the *interaction graph* denoted G_f. It has one vertex per automaton, and an arc from i to j whenever $f_j : \{0, 1\}^n \to \{0, 1\}$ effectively depends on its i-th component (formally, if there exists $x \in \{0, 1\}^n$ such that $f_j(x) \neq f_j(x + e_i)$ with e_i the i-th base vector and addition taken modulo two). The graph G_f captures the architecture of the network, through the mutual influences among automata. Signs may be added to the arcs of G_f, representing two non-exclusive types of influence: positive (resp. negative) when there exists x such that flipping x_i from 0 to 1 flips f_j from 0 to 1 (resp. from 1 to 0).

Past achievments A series of metatheorems has been obtained in recent years, both in the deterministic and the non-deterministic settings. They encompass at once a large range of developments. Standard dynamical system approaches aim at understanding the dynamics $\mathscr{G}_f$, in terms of the local functions $(f_i)_{i\in[n]}$. This naturally turns into decision problems of the form: given $(f_i)_{i\in[n]}$, does $\mathscr{G}_f$ has a given property? In order to give the input, we encode $\mathscr{G}_f$ as a Boolean circuit:

- on $\log_2(X)$ inputs and $\log_2(A_i)$ outputs in the deterministic case,
- on $2\log_2(X)$ inputs and 1 output in the non-deterministic case.

It corresponds to algorithmic descriptions of the behavior, and is equivalent to a *succinct representation* of the graph $\mathscr{G}_f$. Many problems can easily be proven to be NP-complete (e.g. the existence of fixed points, and of limit cycles of a given length) or coNP-complete (e.g. the injectivity of $\mathscr{G}_f$) [2].

Theorem ([3, 4])**.** *Given a deterministic or non-deterministic AN f encoded as a circuit, any cliquewidth-non-trivial question on $\mathscr{G}_f$ expressed in monadic second-order graph logic is* NP*-hard or* coNP*-hard.*

Monadic second-order graph logic (MSO) are formulas evaluated on graphs (finite in our setting), constructed with the usual connectives $\wedge, \vee, \neg, \Rightarrow$ and quantifications $\exists, \forall$ on vertices and sets of vertices. Atoms are build from the signature $\{=, \to\}$ where $x \to y$ is true when there is an arc from x to y. To an MSO formula we associate a decision problem. The non-triviality condition asserts that the formula has an infinity of models of bounded cliquewidth and an infinity of countermodels of bounded cliquewidth. The cliquewidth is not relevant in the deterministic setting, it serves as a technical tool to pump between idenditical MSO-types inside clique-decompositions (applying compositionality). At the end, in order to obtain the theorem above, we proceed to a polynomial time reduction. It constructs, from a **SAT** or **UNSAT** instance (there is some unknown symmetry to deal with), the circuit encoding of an AN's dynamics $\mathscr{G}_f$.

Research objectives We aim at transferring such metatheorems of complexity lower bound to other models of computation, via strict simulations acting as reductions. The complexity of other finite dynamical systems has been characterized, such as (finite) cellular automata [5] and reaction systems [6] (we are interested in basically anything that computes: self-assembly tilings, DNA folding models...) For each new model of computation, two complementary strategies are possible.

1. Reduce from automata networks. This is based on the intuition that if model A simulates model B via $\varphi : B \to A$, meaning that $A^k \circ \varphi = \varphi \circ B$, then $\varphi + A$ is at least as complex as B. If φ is polytime computable, it can act as a reduction, provided that the answer to the decision problem under consideration is preserved. Many families of simulations may be compared:
   - strictly step-by-step ($k = 1$),
   - non-uniformly step-by-step (k varies),
   - minor (subdivised reachability),
   - total (φ surjective),
   - bijective (φ bijective),
   - asymptotic (on the limit dynamics).

   A broader understanding of their relationship, in terms of computational complexity, may help to deepen our understanding of the abstract notion of *simulation*.
2. Implement the last step of the theorem for another model of computation. Here the challenge is to obtain a dynamics with a prescribed clique-decomposition (which is a tree labeled by operations describing how to construct the graph). The pieces of clique-decomposition are simple enough, so that the reduction to automata networks can actually be performed in logarithmic space, which is encouraging [7]. Furthermore, the pumping techniques seems to have room for improvement.

Progresses on automata networks have already been achieved, which may ease the task. In particular, the theorem holds for q-uniform networks (which generalises the Boolean case to any alphabet of size $q \geq 2$ shared by all the automata) [8], and the complexity has been characterized when the input automata network verifies additional restrictions (also expressed in MSO graph logics) [9].

Expected collaborations The local environment is rich, with researchers from LIS teams CANA (natural computation), MOVE (logics and model checking), and DALGO (distributed computing on graphs). Intensive exchanges are of course planned with the co-authors of the first «à la Rice» complexity lower bounds for finite dynamical systems [3, 4], from I2M team GDAC at Marseille Luminy, and from LORIA team MOCQUA in Nancy. Colleagues from I3S team MC3 are also deeply involved in the project. The PhD would take place in an active network of international collaborations around the subject, specifically with Chile (group of Eric Goles) and Brazil (group of Pedro Balbi).

References

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