Projet ANR **ALARICE**

"Bornes de complexité générales pour les systèmes dynamiques finis"

♡ **Théorème** [Rice, 1953]**.** Toute propriété non-triviale du comportement des programmes est indécidable.

↓ Systèmes dynamiques finis donc décidables, donc complexité algorithmique.

Métathéorème [Objectif 1]**.** Toute propriété non-triviale de la dynamique des $\langle$mon_modèle_de_calcul$\rangle$ est $\mathcal{C}$-difficile.

Pour quelles définitions de *propriété non-triviale* et *classe $\mathcal{C}$* ?

Plan

- https://alarice.lis-lab.fr/
  ↪ essayons de la maintenir à jour (publications, évènements) : git push ou mailto:kevin

- Réseaux d'automates et architecture des interactions (EN)
- 3 Objectifs (EN) ⇐ **science et problèmes ouverts !**
- 5 Tâches
- Budget et planning

Automata network $\frac{1}{3}$: definition

$$\begin{aligned} & f &: \{0,1\}^n \to \{0,1\}^n \\ \text{as} \quad & f_i &: \{0,1\}^n \to \{0,1\} \quad \text{for } i \in [n] \end{aligned}$$

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Local functions f_i

Dynamics $\mathscr{G}_f$

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$n = 4$

$f_1(x) = x_1$

$f_2(x) = x_2$

$f_3(x) = x_3$

$f_4(x) = \varphi(x_1, x_2, x_3) \vee \neg x_4$

$$\varphi(x_1, x_2, x_3) = \neg[(x_1 \vee x_2) \Rightarrow \neg(x_2 \wedge x_3)] \equiv x_2 \wedge x_3$$

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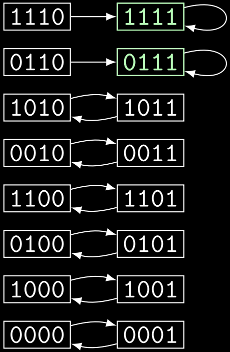

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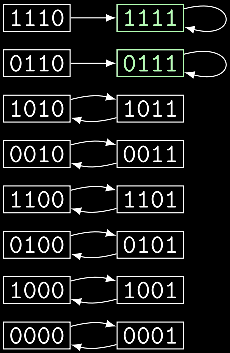

Theorem. Given f, it is NP-complete to decide if f has a fixed-point.

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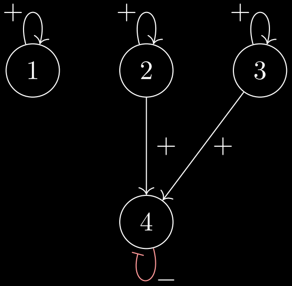
Interaction graph G_f



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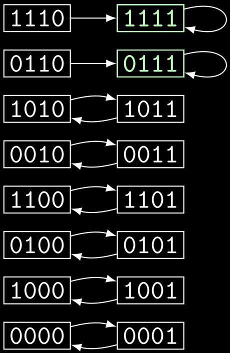

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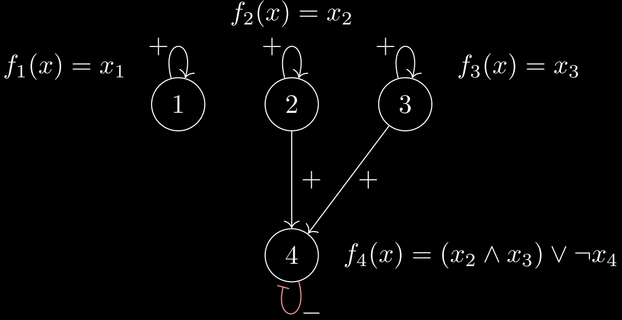
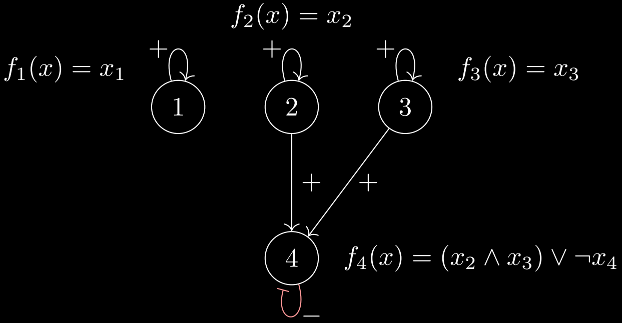


Automata network $\frac{2}{3}$: interaction graph (architecture)

$$(i,j) \in G_f \iff \exists x \in \{0,1\}^n : f_j(x) \neq f_j(x+e_i)$$

$+$ when $f_j(x) = x_i$
$-$ when $f_j(x) \neq x_i$

possible to have $\pm$ arcs



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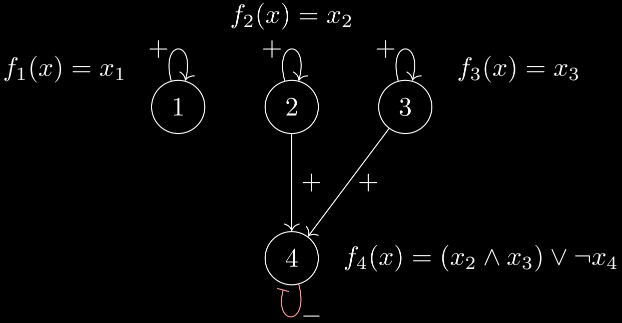
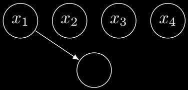


Also
- non-Boolean alphabets
- other update schedules (e.g. non-deterministic)

Automata network $\frac{3}{3}$: encoding f by circuits



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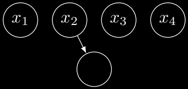

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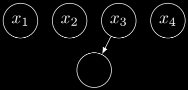

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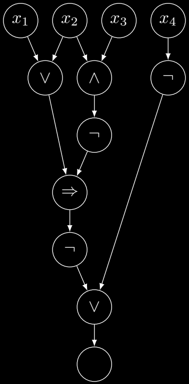

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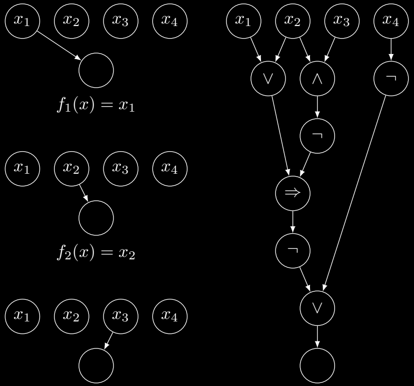

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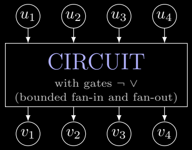
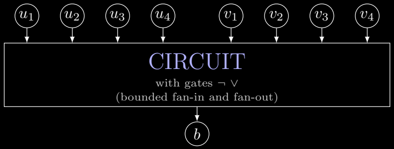
Succinct graph representation of $\mathscr{G}_f$

$\hookrightarrow$ config/vertex label on $\log(|V|)$ bits

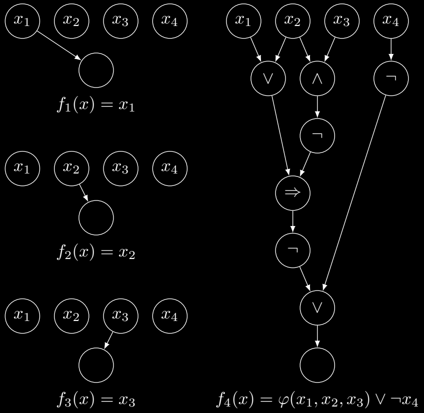
Deterministic



Non-deterministic



Automata network $\frac{3}{3}$: encoding f by circuits



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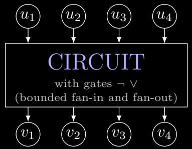

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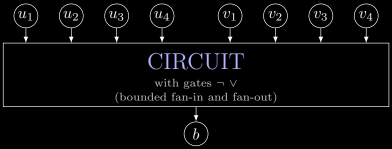


Property. Any (functional) digraph is the dynamics of some AN.

Proof. With one automaton of alphabet V and $f_1(x) = \{y \in V \mid (x, y) \in E\}$. □

Caution. An AN with n automaton of alphabet $\{0, \ldots, q-1\}$, *i.e.* a q-uniform AN, has $|\mathscr{G}_f| = q^n$. Boolean is $q = 2$.

3 Objectifs

$$[n] = \{1, 2, \ldots, n\}$$

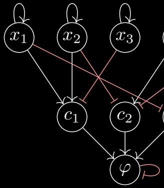
$$f_i : \{0,1\}^n \to \{0,1\} \text{ for } i \in [n]$$

Graphe d'interaction G_f sur les sommets $[n]$

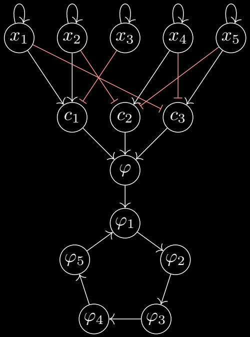
Dynamique $\mathscr{G}_f$ sur les sommets $\{0,1\}^n$

- ▷ Objectif 1 : atteindre des métathéorèmes
- ▷ Objectif 2 : épuiser les problèmes classiques
- ▷ Objectif 3 : transposer à d'autres modèles de calcul

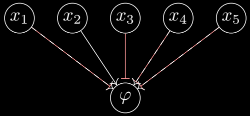
Objective 1. Metatheorems $\frac{1}{5}$: go general



$\exists x : x \to x$
NP-complete



$$\begin{aligned}\exists x^1, \ldots, x^k : x^1 &\to x^2 \\ \wedge\ x^2 &\to x^3 \\ \wedge\ \ldots& \\ \wedge\ x^k &\to x^1\end{aligned}$$
NP-complete



$\exists x : \forall y : y \to x$
coNP-complete

$\forall x : \neg(x \to x)$
coNP-complete

$\exists x, y : x \to y$
$\mathcal{O}(1)$

Objective 1. Metatheorems $\frac{2}{5}$: statement « *à la Rice* » ♡

Metatheorem. Given f, any nontrivial property of $\mathscr{G}_f$ is hard.

- ▷ Property "Graph FO".

- ▷ Nontrivial.

- ▷ Hard.

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- Property "Graph FO". $\neg, \wedge, \vee, \Rightarrow, \overset{\textit{configs}}{\exists, \forall}$ on signature $\{=, \to\}$
  $\exists x \forall y : (y \to x) \vee (x = y)$

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$$\exists x : x \to x \qquad \text{fixed-point}$$
$$\exists x^1, x^2, x^3 : (x^1 \to x^2) \wedge (x^2 \to x^3) \wedge (x^3 \to x^1) \qquad \text{3-cycle}$$
$$\forall x, x', y : [(x \to y) \wedge (x' \to y)] \Rightarrow (x = x') \qquad \text{injectivity}$$

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> *Input :* the circuits of an automata network f.
> *Ouput :* does $\mathscr{G}_f \models \psi$?

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Theorem [GGPT 2021]. If ψ is nontrivial then **ψ-dynamics** is NP-hard or coNP-hard, otherwise it is $\mathcal{O}(1)$.

FO, deterministic

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now MSO and q-uniform (SGRs → ANs) for any $q \geq 2$ $\rightsquigarrow$ Aliénor !
[GGPT 2023]
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Deterministic: clear trivial versus nontrivial dichotomy.
$\Longrightarrow$ $\mathcal{O}(1)$ versus NP-hard or coNP-hard.

Non-deterministic: arbitrary graphs

- arborescent (nontrivial on bounded treewidth) q-arborescent implies NP-hard or coNP-hard.

  (ψ arborescent iff $\exists k$ such that ψ has an infinity of models and countermodels of treewidth $\leq k$)

  $\hookrightarrow$ a succinct counterpoint to Courcelle's theorem 😎
  $\hookrightarrow$ we need a bounded structure to pump... cw ? $\rightsquigarrow$ Pierre O !

- no parameter fails: $\exists\psi$ FO such that if **SAT** or **UNSAT** reduces to ψ-**dynamics** then polytime algo for **SAT** on a robust subset.

  (M robust iff $\forall k : \exists \ell : \{\ell, \ell+1, \ldots, \ell^k\} \subset M$ [Fortnow Santhanam 2017])

Objective 1. Metatheorems $\frac{4}{5}$: pump and reduce

Metareduction from SAT (or UNSAT).

1. For any nontrivial ψ, finite model theory gives G_0, G_1, G_2, G_3 with:

$$G_2 \oplus G_1 \oplus G_1 \oplus \cdots \oplus \ G_1 \ \oplus \cdots \oplus G_1 \oplus G_1 \oplus G_3 \not\models \psi$$

$$G_2 \oplus G_1 \oplus G_1 \oplus \cdots \oplus \ G_0 \ \oplus \cdots \oplus G_1 \oplus G_1 \oplus G_3 \models \psi$$

We can assume $|G_1| = |G_0|$.

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2. For simplicity, assume $G_2 = G_3 = \emptyset$ and $\oplus = \sqcup$.
   Given φ on n variables we construct f on $n+1$ automata.
   f evaluates φ on the n first Boolean variables, and:
   - ▷ if φ is not satisfied then we produce in $\mathscr{G}_f$ a copy of G_1 using the last automaton of alphabet $\{1, \ldots, |G_1|\}$
   - ▷ if φ is satisfied then we produce in $\mathscr{G}_f$ a copy of G_0 using the last automaton of alphabet $\{1, \ldots, |G_0|\}$

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Remark. G_0 depends only on the quantifier rank of ψ,
$G_0 \models \psi$ gives NP-hard, $G_0 \not\models \psi$ gives coNP-hard,
and to get G_1 we **pump** on the other infinity. (structural bound)

Objective 1. Metatheorems $\frac{5}{5}$: perspectives

- Most comprehensive structural bounds to pump ?
  What comes after cliquewidth ?
- Reduce from other problems ?
  NP as non-deterministic circuit evaluation...
- Higher levels of PH ? Connectivity is PSPACE-complete...
  For any i there is ψ_i such that ψ_i-**dynamics** is Σ_i^{P}-complete.
- Enrich the signature $\{=, \rightarrow\}$ to distinguish configurations ?
  Unary relation $S_i^a(x)$ for "$x_i = a$" gives P-complete formulas...

Objective 2. Classic $\frac{1}{4}$: on input G_f

$\{\emptyset, +, -, \pm\}$

On n automata, there are 2^{n2^n} Boolean networks and 4^{n^2} signed digraph.

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Literature: structural bounds

Theorem [Riis 2007, Aracena 2008, Aracena et al. 2017].

- Unsigned $\texttt{max} \leq 2^{\texttt{minFAS}}$
- Unsigned monotonous $\texttt{maxCyclePacking} + 1 \leq \texttt{min}$
- Signed $\texttt{max} \leq 2^{\texttt{min+FAS}}$
- Signed monotonous $2^{\texttt{maxCyclePacking*}} \leq \texttt{min}$

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Théorème [BDPR 2019,2022]. Given a signed interaction graph G, deciding...

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↪ Explains why structural bounds are loose...

Objective 2. Classic $\frac{1}{4}$: on input G_f

On n automata, there are 2^{n2^n} Boolean networks and 4^{n^2} signed digraph. $\{\emptyset, +, -, \pm\}$

Literature: structural bounds

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Perspectives. Unsigned ? Limit-cycles ? non-Boolean ?

Objective 2. Classic $\frac{2}{4}$: compute the interaction graph G_f

Given $f : \{0,1\}^n \to \{0,1\}^n$ as a circuit and G, does $G_f = G$?

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Effectively non-trivial graph properties are NP-hard or coNP-hard:

1. $\exists$ Polytime algorithm A deciding whether K_n has the property,
2. $\exists$ Polytime algorithm B producing G, G' on n vertices such that G has the property and G' does not.

Objective 2. Classic $\frac{3}{4}$: $\mathscr{G}_f$ up to isomorphism

What can be said on $G(\mathscr{G}) = \{G_f \mid \mathscr{G}_f \sim \mathscr{G}\}$ from $\mathscr{G}$ on $\{0,1\}^n$?

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Theorem [BPPMR 2023]. $K_n \in G(\mathscr{G})$ for all $\mathscr{G}$, except :

- the identity $\forall x : f(x) = x$ (G_f has all loops)
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Theorem [BPMR 2024]. There are universal dynamics :

- $\exists \mathscr{G} : G(\mathscr{G})$ contains all digraphs on $[n]$ except $G_\emptyset$, with big alphabet.
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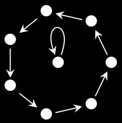

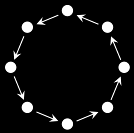

$\rightsquigarrow$ Florian !

[ABGPRT 2023]. $\Delta(G_f) \leq 2$ ✓ Open. Bounded degree ?

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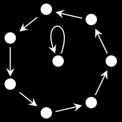

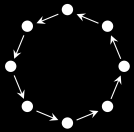

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Theorem [Robert 1980]. G_f acyclic implies f^n constant.

Open. What kinds of $\mathscr{G}$ up to iso have an acyclic graph in $G(\mathscr{G})$?

Objective 2. Classic $\frac{4}{4}$: update modes

A vast world !

Objective 2. Classic $\frac{4}{4}$: update modes

- ▷ Block-sequential. $\mathtt{BS}_n$ e.g. $(\{1,2,3\},\{6\},\{4,5\})$

  Theorem [Robert 1986]. Fixed-point invariance.

  The update schedule itself can embed complexity :

  Theorem [BGMPS 2021]. Given $f : \{0,1\}^n \to \{0,1\}^n$ as a circuit :

  - $\exists \beta \in \mathtt{BS}_n$: f_β has a k-limit-cycle ? is NP-complete.
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▷ Block-parallel. BP_n e.g. $\{(1,2,3),(6),(4,5)\}$

gives $(\{1,6,4\},\{2,6,5\},\{3,6,4\},\{1,6,5\},\{2,6,4\},\{3,6,5\})$

It has *lcm* substeps.

Theorem [PST 2024]. Given $f : \{0,1\}^n \to \{0,1\}^n$ as a circuit and $\mu \in \mathrm{BP}_n$, almost all problems become PSPACE-complete : image, $\exists$ fixed-point, $\exists$ k-limit-cycle, Subdynamics-G, reachability, constant, but bijectivity is still coNP-complete.

Open. Complexity of recognizing the identity ?

Observation. BP_n's repetitions can generate new fixed points. ⇝ Léah !

Objective 3. Other models $\frac{1}{2}$: simulation as reduction

Goal. Transfer Rice-like complexity lower bounds to:

- finite cellular automata
- reaction systems
- tile assembly
- DNA folding
- …

via simulations acting as reductions.

Intuition. If A simulates B via $\varphi : X_B \to X_A$ $\quad A^k \circ \varphi = \varphi \circ B$
then $\varphi + A$ is at least as complex as B.

- strictly step-by-step ($k = 1$)
- non-uniformly step-by-step (k varies)
- minor (subdivised reachability)
- total (φ surjective)
- bijective (φ bijective)
- asymptotic (on the limit dynamics)

$\}\rightsquigarrow$ Simulation thesis !

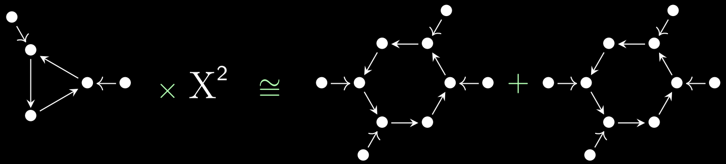
Objective 3. Other models $\frac{2}{2}$: alternative-parallel algebra

Abstract finite deterministic dynamical system (FDDS)
are functional digraphs up to isomorphism.

Decomposition algebra (commutative semiring):

- $+$: alternative execution (disjoint union)
- $\times$: parallel execution (direct product)

Goal. Classify the complexity of solving equations. $\rightsquigarrow$ Enrico & Marius !



5 Tâches

- Tâche 0 : Coordination (rapports...). Adrien, Kévin
- Tâche 1 : Objectif 1 Metatheorems. Guillaume, Kévin
- Tâche 2 : Objectif 2 Classic. Adrien, Sylvain
- Tâche 3 : Objectif 3 Other Models. Enrico, Sara
- Tâche 4 : Diffusion (papiers, exposés, vulgarisation). Florian, Kévin, Pierre

Budget et planning $\frac{1}{2}$

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Extra slide : bounding $\texttt{pfmax}(G)$ $\frac{1}{3}$: problem

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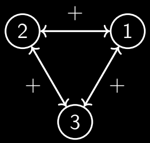
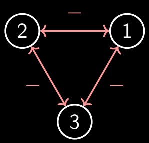


$\mathtt{max} = 2$

$\mathtt{min} = 2$

8 networks

1 2 3	∧∧∧	∨∧∧	∨∨∧	∨∨∨
000	000	000	000	000
001	000	100	110	110
010	000	100	100	101
011	100	100	110	111
100	000	000	010	011
101	010	110	110	111
110	001	101	111	111
111	111	111	111	111



$\mathtt{max} = 3$

$\mathtt{min} = 1$

8 networks

1 2 3	∧∧∧	∨∧∧	∨∨∧	∨∨∨
000	111	111	111	111
001	001	101	111	111
010	010	110	110	111
011	000	000	010	011
100	100	100	110	111
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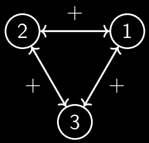

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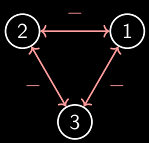

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$\{\varnothing, +, -, \pm\}$

Number of effective monotone local functions of degree 0,1,2,3,4,5,6,7,8:
2, 1, 2, 9, 114, 6894, 7785062, 2414627396434, 56130437209370320359966 (OEIS/A006126)

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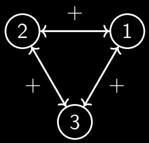
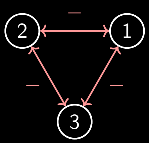


$\texttt{max} = 2$
$\texttt{min} = 2$
8 networks



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Fixed-points

♡

Positive cycles
(even number of — arcs)

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- Lemma [$\Longrightarrow$ by Aracena 2008].
  $\texttt{max}(G) \geq 1 \iff$ each initial strongly connected component of G has a positive cycle.

- Theorem [Robertson, Seymour, Thomas 1999; McCuaig 2004]. ⚠
  We can decide in polytime whether a given graph has a positive cycle.

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Proof sketch. $\texttt{max}$ $k = 1$ $k = 2$

- Upper bound NP: not trivial because checking $G_f = G$ is DP-complete.
- Lower bound NP: reduction from SAT. $\varphi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$

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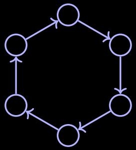
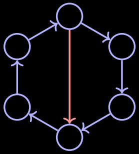
Proof sketch. $\texttt{max}$ $k = 1$ $k = 2$

- Upper bound NP: not trivial because checking $G_f = G$ is DP-complete.
- Lower bound NP: reduction from SAT. $\varphi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$
  Basic observation.



2 fixed points



1 fixed point

The idea is to "neutralize" such negative chords by satisfying φ.

Extra slide : bounding pfmax(G) $\frac{2}{3}$: results

Theorem [Bridoux, Durbec, P, Richard 2019,2022]. Given a signed G, deciding...

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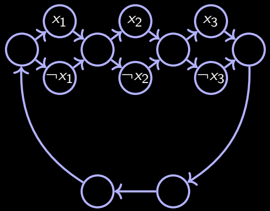

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Extra slide : bounding $\texttt{pfmax}(G)$ $\frac{2}{3}$: results

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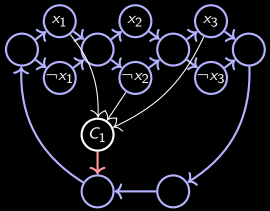

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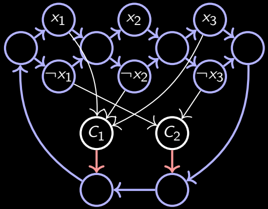

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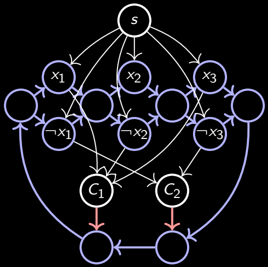

In order to get two fixed-points $x \neq y$:

1. Each clause must be "neutralized" by a literal equal in both fixed-points.

But never $x_i = y_i$ and $\neg x_i = \neg y_i$ because:

2. Distinct fixed-points must differ on a positive cycle.

Extra slide : bounding $\texttt{pfmax}(G)$ $\frac{3}{3}$: proof ≥ 2

Theorem. $\texttt{pfmax}(G) \geq 2$ is NP-hard.

Proof sketch. Reduction from **3SAT**
Given ψ we construct G_ψ such that ψ is satisfiable $\iff$ $\texttt{pfmax}(G_\psi) \geq 2$

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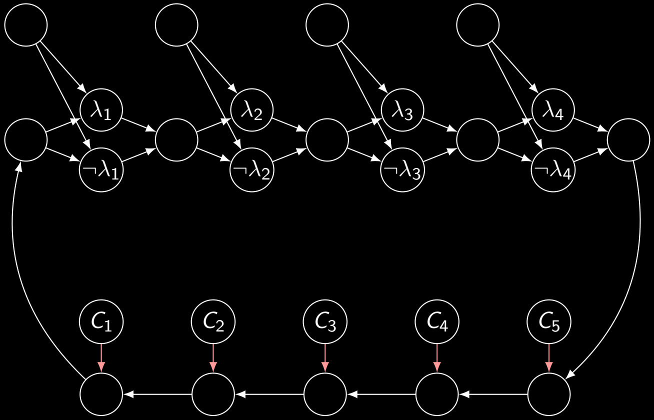

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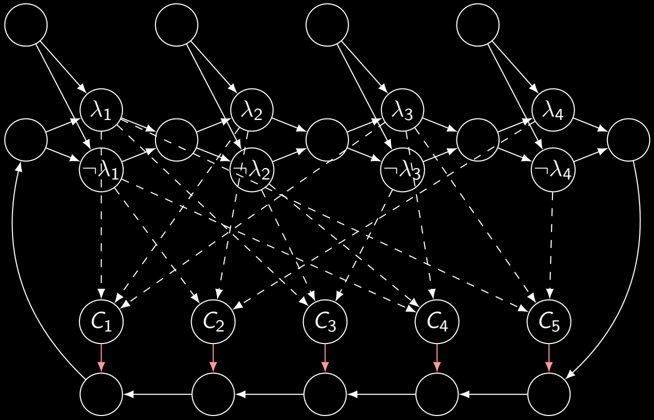

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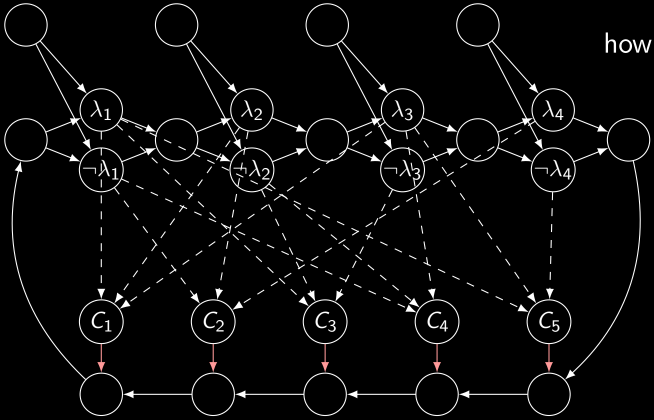

how to get $x \neq y$?

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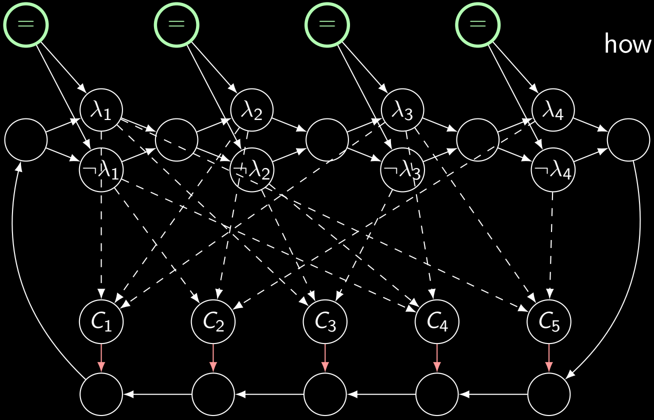

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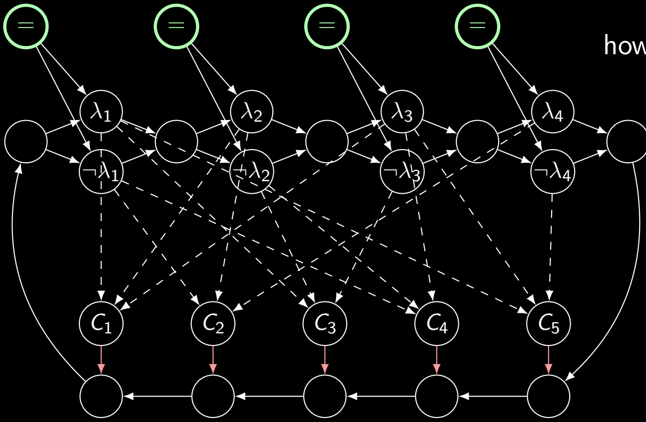

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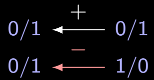

coherent+ in-neighbor

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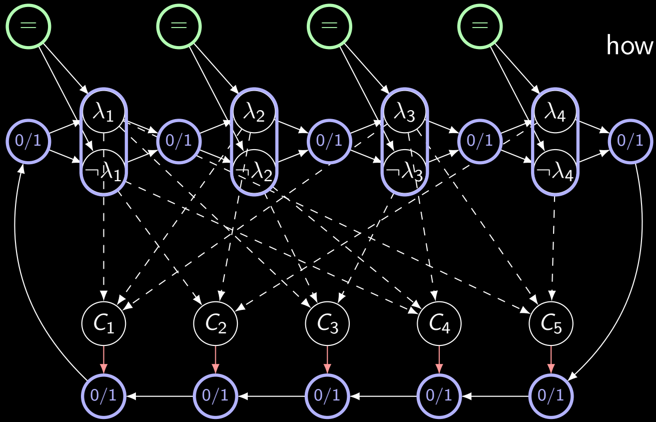

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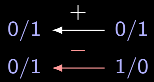

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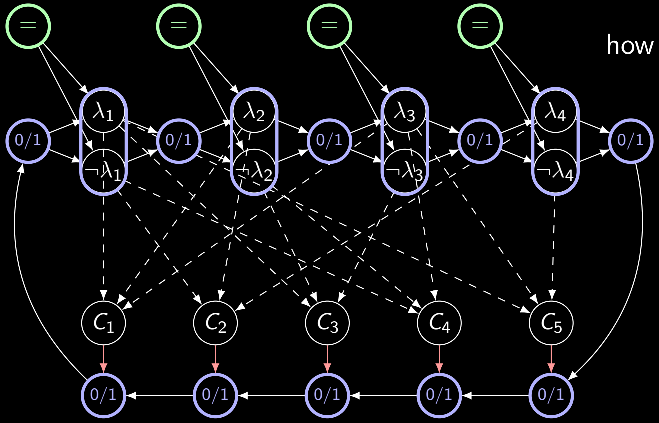

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   coherent+ in-neighbor
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   0/1 ? 1/0 ?
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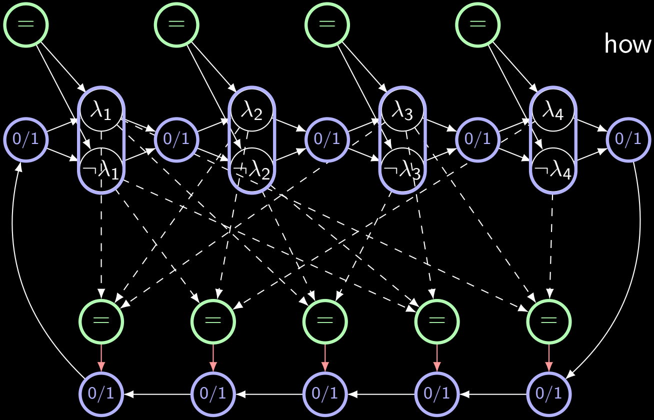

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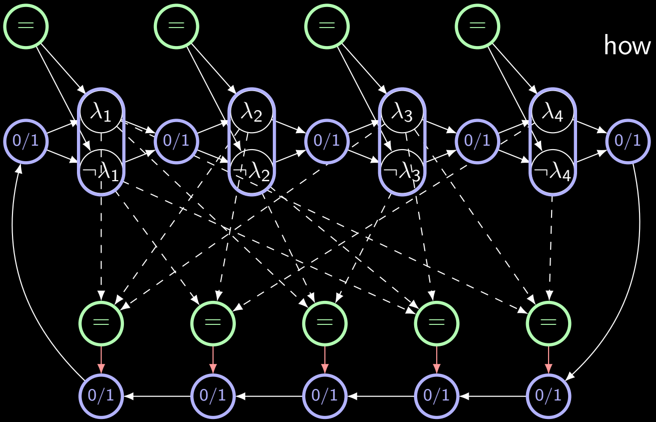

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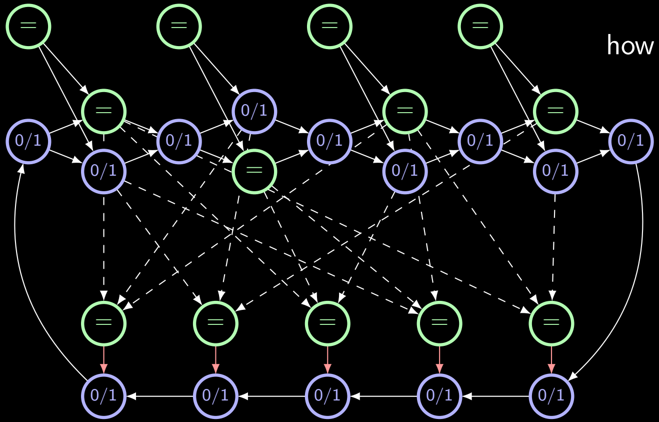

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$\iff$ ψ is satisfiable

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