FO satisfiability for cellular automata over finite graphs

Coup d'envoi ALARICE

G. Theyssier

Institut de mathématiques de Marseille
(CNRS, Université Aix-Marseille)

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- ϕ**-SAT problem** for fixed property ϕ
  - *input:* **local rule** λ
  - *question:* $\exists$? graph *G such that* $\texttt{dynamics}(G, \lambda) \models \phi$

Why?

- automata network theory
  - intrinsic universality (work with M. Rios-Wilson)
  - interaction/communication graphs
- T. Tao's universality program on PDE from physics
  - potential well dynamics
    `https://arxiv.org/abs/1707.02389`
  - incompressible Euler equation on compact manifolds
    `https://arxiv.org/abs/1902.06313`
- finite model theory: undecidable thresholds in MSO fragments

A basic model

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- **properties**: FO formulas on orbits
  - example: surjectivity $\stackrel{def}{=} \forall x \exists y, y \to x$
  - $\phi \rightsquigarrow$ problem ϕ-SAT

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Theorem

For any MSO1 formula Ψ there are λ and ϕ such that

$$\mathcal{G}(\Psi) = \mathcal{G}(\lambda, \phi)$$

and $\Psi \mapsto (\lambda, \phi)$ is computable.

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Theorem bis

For any MSO1 formula Ψ there are λ and ϕ such that

1. $\mathcal{G}_{con}(\Psi) = \mathcal{G}_{con}(\lambda, \phi)$,
2. ϕ only depends on **prefix signature** of prenex Ψ,

and $\Psi \mapsto (\lambda, \phi)$ is computable.

- **prefix signature:** $\exists X, \forall Y, \forall x, \forall y, \forall z \rightsquigarrow \exists_2 \forall_2 \forall_1$
- from now on, all graphs are connected!

Undecidable ϕ-SAT

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- Turing machine T $\mapsto \Psi_T$ MSO formula s.t. $G \models \Psi_T$ iff
  "G is a grid that can hold an halting space-time diagram of T"
- check that prefix signature of Ψ_T is independent of T
- apply "theorem bis": $\Psi_T \mapsto (\lambda_T, \phi_T)$
- ϕ_T does **not** depend on T!
- So there is a fixed ϕ such that:

$$\lambda_T \in \phi\text{-SAT} \iff \mathcal{G}(\lambda_T, \phi) \neq \emptyset \iff \mathcal{G}(\Psi_T) \neq \emptyset \iff T \text{ halts}$$

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Question 1

The smallest ϕ with ϕ-SAT undecidable?

BOO USA!

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- $Q = \{0, 1\}$
- $S_k(c, v) = \{c_{v'} : d(v', v) = k\}$
- $\lambda : \{0, 1\} \times \left(2^{\{0,1\}}\right)^R \to \{0, 1\}$ of radius R:

$$F(c)_v = \lambda\big(c_v, S_1(c, v), S_2(c, v), \ldots, S_R(c, v)\big)$$

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Questions 2

- is there an undecidable ϕ-SAT in this model?
- can we capture all MSO?
- what about larger (fixed) alphabet?

Back to classical FO SAT

- $\mathcal{F}$: family of dynamical systems
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Theorem (unpublished but true ☺)

There is a 2D CA F such that FO-SAT is **undecidable** for $\mathcal{F} = \{F\}$.

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Questions 3

- $\mathcal{F} = \{q$-uniform automata networks$\}$?
- $\mathcal{F} = \{$Boolean networks$\}$?
- $\mathcal{F} = \{F_{G,\lambda} : G$ finite graph$\}$ for some fixed λ

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Endless variations

- replace sets by (capped) multisets: $2^Q \rightsquigarrow [k]^Q$
- **non-uniform CA**: add labels on vertices
- **Cayley graphs**: add labels on edges + λ distinguishes incoming neighbors
- undecidability for other properties?
  - **intrinsic universality**
  - producing large cycles/transients
- replace SAT by ω-nontrivial