Some open problems – ANR **ALARICE**

"General complexity bounds for finite dynamical systems"

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Automata networks (model any finite discrete dynamics)

- ▷ size n, automata $[n] = \{1, \ldots, n\}$, alphabet $[\![q]\!] = \{0, \ldots, q-1\}$
- ▷ configurations $X = \Pi_{i\in[n]} A_i$ with $A_i = [\![q_i]\!]$ with $q_i \in \mathbb{N}_+$ for all $i \in [n]$
- ▷ (AN) deterministic automata networks $f : X \to X$ split into $f_i : X \to A_i$ for each $i \in [n]$
- ▷ (NAN) nondeterministic automata networks $f \subseteq X \times X$
- ▷ Boolean $X = \{0, 1\}^n$ (considered by default), q-uniform $X = [\![q]\!]^n$
- ▷ dynamics $\mathscr{G}_f$ on vertices X (the graph of the relation f, deterministic has out-degree 1)
- ▷ interaction digraph G_f on vertices $[n]$ with arcs $(i, j) \in G_f \iff \exists x : \; f_j(x+e_i) \neq f_j(x)$
- ▷ graph FO and MSO formulas on signature $\{=, \to\}$, denote $\mathscr{G}_f \models \psi$ when $\mathscr{G}_f$ verifies ψ
  Quantify on (sets of) configs, *e.g.* $\forall x, x', y : [(x \to y) \wedge (x' \to y)] \Rightarrow (x = x')$ asks injectivity
- ▷ circuit encoding of a deterministic $f : X \to X$ has $\lceil \log_2(X) \rceil$ input bits and $\lceil \log_2(X) \rceil$ output bits
- ▷ circuit encoding of a nondeterministic $f \subseteq X \times X$ has $2 \cdot \lceil \log_2(X) \rceil$ input bits and 1 output bit
- ▷ withtout restriction on the alphabet, these encodings are succinct graph representations (SGR)
- ▷ update modes are only considered in a dedicated subsection (elsewhere, deterministic = parallel)

1 Unveil metatheorems

Here the structures are directed graphs representing the dynamics of some AN or NAN.
ψ is *q-nontrivial* iff ψ has an infinity of models and of countermodels among q-uniform networks.
ψ is *q-arborescent* iff there exist k such that ψ has an infinity of models and of countermodels of treewidth $\leq k$ among q-uniform networks.
All results hold identically with or without the restriction to q-uniform networks (ask Aliénor).

ψ-AN-dynamics	**ψ-NAN-dynamics**
Input : the circuit of a deterministic AN f.	*Input :* the circuit of a nondeterministic NAN f.
Ouput : does $\mathscr{G}_f \models \psi$?	*Ouput :* does $\mathscr{G}_f \models \psi$?

Deterministic. For any $q \geq 2$ and MSO formula ψ, either it is q-trivial and **ψ-AN-dynamics** is solvable in time $\mathcal{O}(1)$, or it is q-nontrivial and ψ-AN-dynamics is NP-hard or coNP-hard. [6, 7]

Nondeterministic. For any $q \geq 2$, if ψ is a q-arborescent MSO formula, then **ψ-NAN-dynamics** is NP-hard or coNP-hard. Currently: we replace treewidth by cliquewidth. No parameter fails: there is ψ FO such that if **SAT** or **UNSAT** reduces to **ψ-NAN-dynamics** then there is a polytime algo for **SAT** on a robust set of instance sizes (M is *robust* iff $\forall k : \exists \ell : \{\ell, \ell+1, \ldots, \ell^k\} \subset M$), which is unlikely. [5, 7]

Saturating. For any m, there is a graph Ω_m such that for any MSO formula ψ of quantifier rank m, either for every graph G we have $G \sqcup \Omega_m \models \psi$, or for every graph G we have $G \sqcup \Omega_m \not\models \psi$.

Perspectives. A quest to expend Rice-like complexity bounds toward dichotomies.

- ▷ We pump for **SAT**, but can we reduce from other problems with other ingredients ?
- ▷ Rice-like lower bounds for higher levels of PH ? **Known.** For any i there is ψ_i such that **ψ_i-AN-dynamics** is Σ_i^{P}-complete [6], and succinct-Connectivity is PSPACE-complete.
- ▷ Enrich the signature $\{=, \to\}$ to distinguish configurations ? Unary relation $S_i^a(x)$ for "$x_i = a$" gives P-complete formulas, same for total/partial order $\leq$, Aliénor showed logspace reduction.

2 Exhaust typical problems

▷ **On input G_f.** It required tricky reductions to prove that, on input a signed G_f (parallel Boolean) [2]: deciding **lower bounds** on the **maximum** number of fixed-points is NP-complete (except $k = 1$ in P), deciding **upper bounds** on the **minimum** number of fixed-points is NEXPTIME-complete (all $k \geq 1$).
Perspectives. Unsigned G_f ? Limit-cycles (requires structural understanding) ? q-uniform for $q > 2$?

- ▷ **Compute G_f.** Given f as a circuit (Boolean AN): deciding $(i,j) \in G_f$ is NP-complete, deciding $(i,j) \notin G_f$ is coNP-complete, and deciding whether G_f equals some given G is DP-complete (even for a single $f_j : \{0,1\}^n \to \{0,1\}$) with $\mathsf{DP} = \{L_1 \cap L_2 \mid L_1 \in \mathsf{NP} \text{ and } L_2 \in \mathsf{coNP}\}$. Furthermore, with the promise $\Delta(G_f) \leq d$ one can compute G_f in polytime. [8] **Open.** Tetrachotomy theorem (P,NP,coNP,DP) ?
- ▷ **$\mathcal{G}_f$ up to isomorphism.** What can be said on $G(\mathcal{G}) = \{G_f \mid \mathcal{G}_f \sim \mathcal{G}\}$ from $\mathcal{G}$ on $\{0,1\}^n$ or on $[\![q]\!]^n$?
  The complete interaction graph on $[n]$ is **universal** *i.e.* $K_n \in G(\mathcal{G})$ for all $\mathcal{G}$ (except the identity and constant), and conversely: **1)** there exists a universal dynamics with big alphabet containing all digraphs (except the empty graph), and **2)** there exists an asymptotically universal dynamics for all $q \geq 3$ (containing a fraction 1 of all digraphs, as $n \to \infty$). [3, 4] **Open.** Boolean ?
  Open. Is it possible to have a bounded-degree Hamiltonian dynamics (limit-cycle of size q^n) ? [1]
  Open. Characterize the $\mathcal{G}$ up to iso having an acyclic graph in $G(\mathcal{G})$? (G_f acyclic implies f^n constant.)
- ▷ **Update modes.** Block-parallel *e.g.* $\{(1,2,3),(6),(4,5)\}$ gives $(\{1,6,4\},\{2,6,5\},\{3,6,4\},\{1,6,5\},\{2,6,4\},\{3,6,5\})$, with lcm substeps. Given an AN f as a circuit and $\mu \in \mathsf{BP}_n$, almost all problems on $f_\mu : \{0,1\}^n \to \{0,1\}^n$ become PSPACE-complete : image, $\exists$ fixed-point, $\exists$ k-limit-cycle, Subdynamics-G, reachability, **constant**… but **bijectivity** is still coNP-complete. **Open.** Complexity of recognizing the **identity** ? [9]

3 Go beyond automata networks

- ▷ **Simulation as reduction.** The goal is to transfer Rice-like complexity lower bounds to finite cellular automata and other models, via simulations acting as reductions. Intuitively, if A **simulates** B via $\varphi : X_B \to X_A$, *i.e.* $A^k \circ \varphi = \varphi \circ B$, then $\varphi + A$ is at least as complex as B. **Simulation thesis** in progress…
- ▷ **Algebra of finite dynamical systems.** Functional digraphs with disjoint-union as + and direct-product as × is a commutative semiring. Solving equations is undecidable in general, but cases are known to be NP-complete and others in P. **Open.** Complexity of division, *i.e.* $AX = B$? [10]

Bonus

The FO question **Determinism**: given f, does $\mathcal{G}_f \models \forall x, y, y' : [(x \to y) \wedge (x \to y')] \Rightarrow (y = y')$?
is polytime equivalent to **$\forall\exists!$-SAT**: given $\varphi(x,y)$, does $\forall x : \exists! y : \varphi(x,y) = \top$?
Open. are they $\mathsf{coNP}^{\mathsf{US}}$-complete ?
Known. $\exists!$-SAT is US-complete ; $\mathsf{coNP} \subseteq \mathsf{US}$; $\mathsf{coNP}^{\mathsf{US}} = \mathsf{coNP}^{\mathsf{NP}}$; **$\exists\exists!$-SAT** is $\mathsf{NP}^{\mathsf{NP}}$-complete. [8]

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