L’algèbre des systèmes dynamiques discrets abstraits

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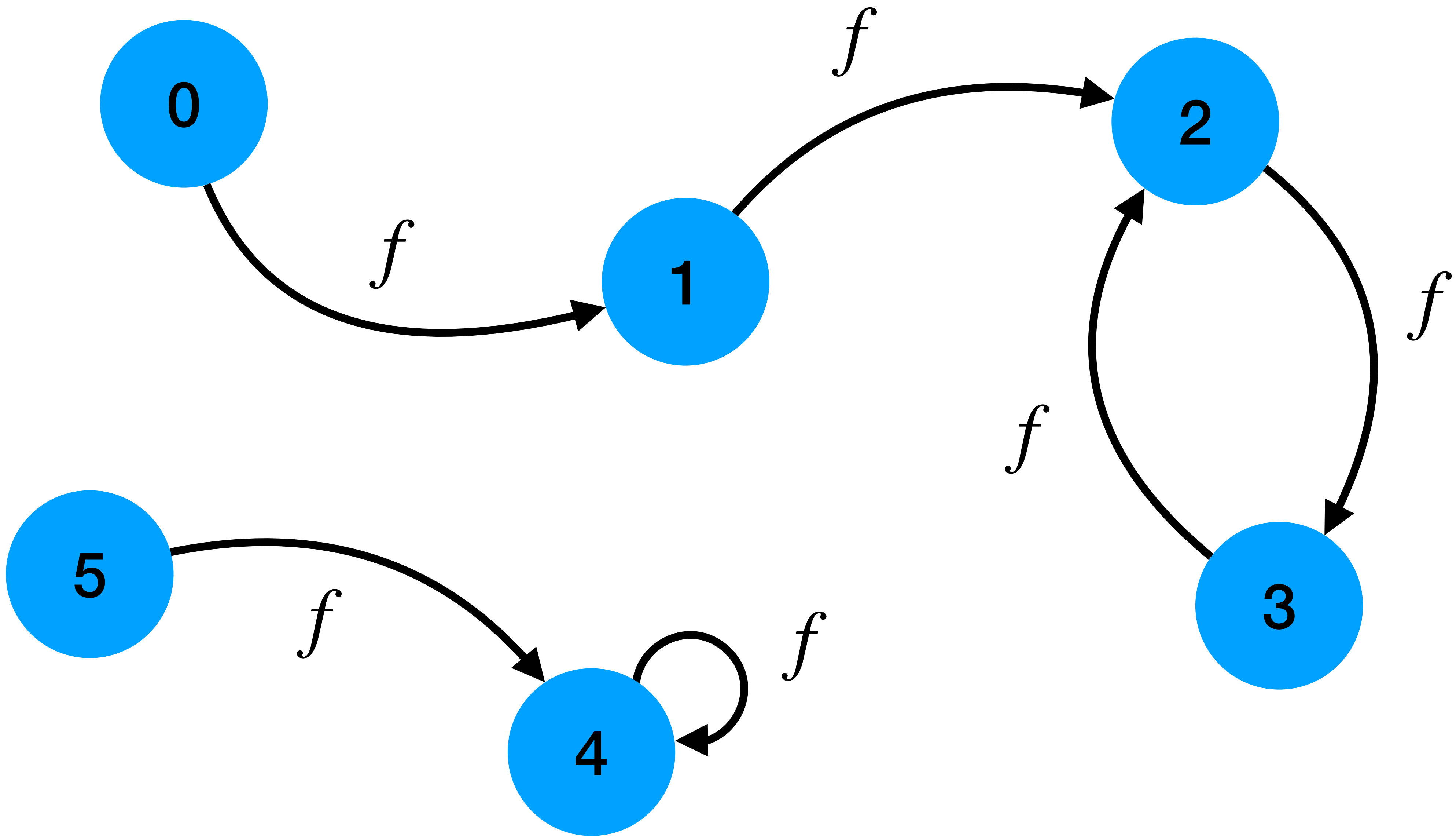
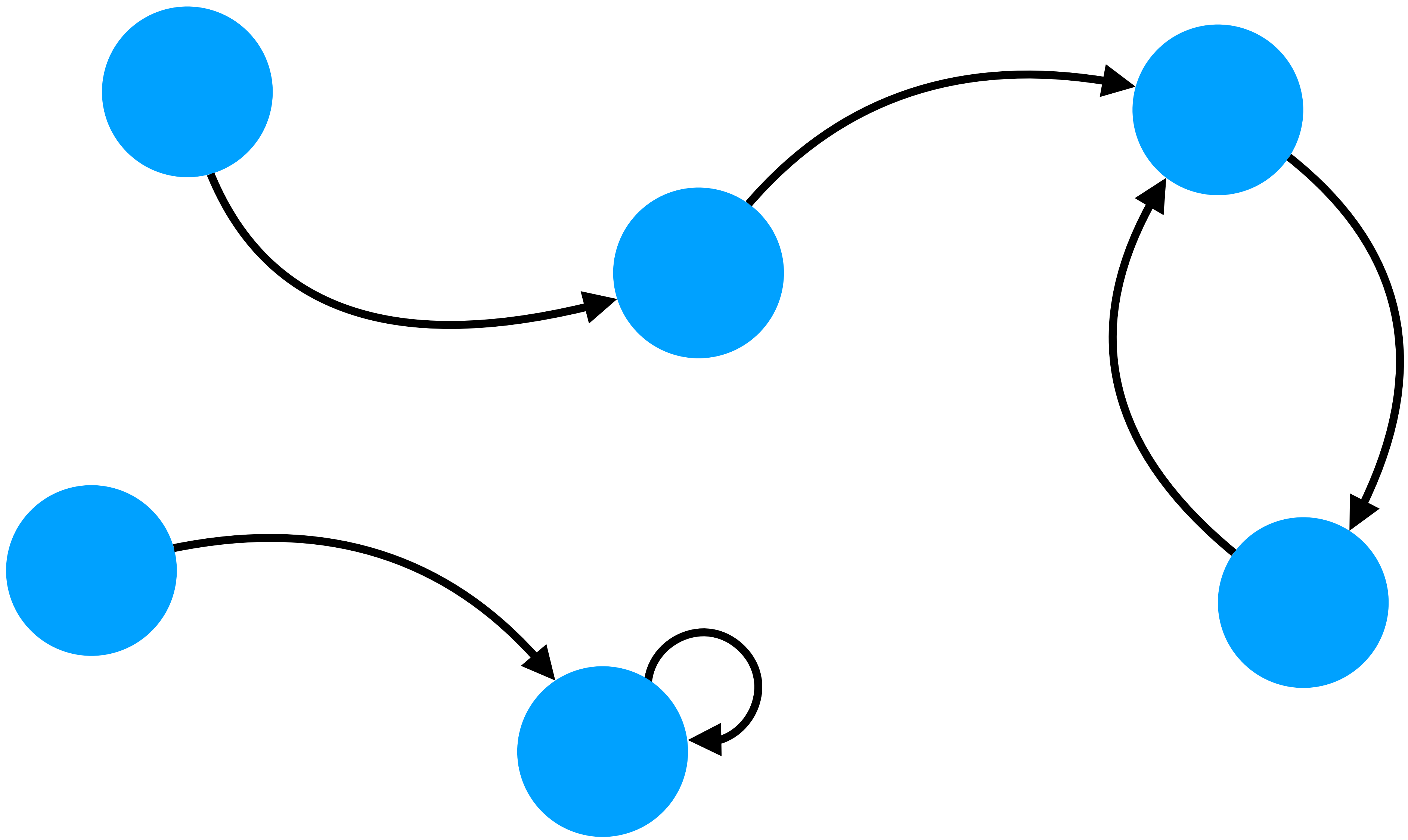
Finite, discrete-time dynamical systems

Just a finite set with a transition function (A, f)



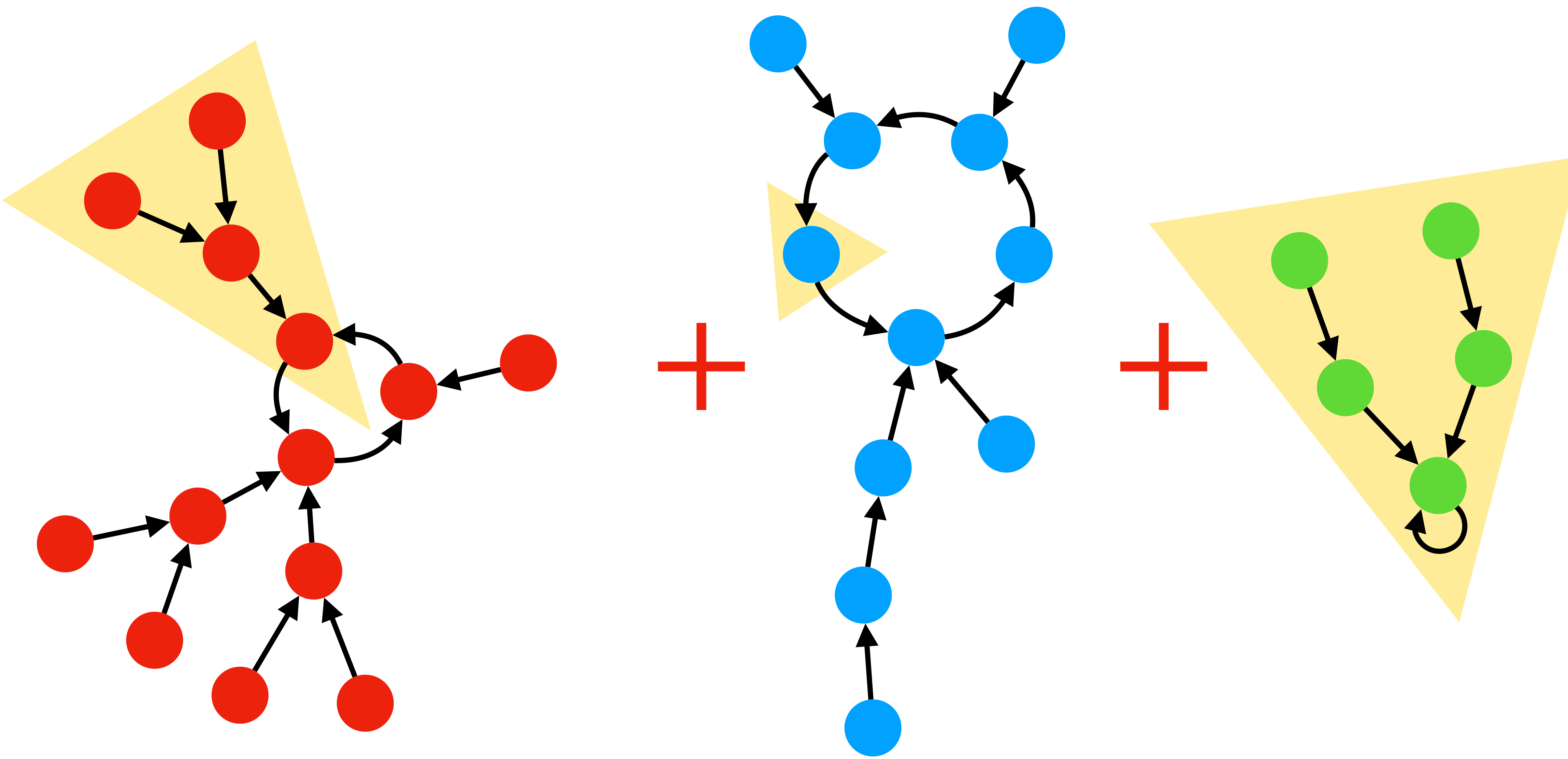
Finite, discrete-time dynamical systems

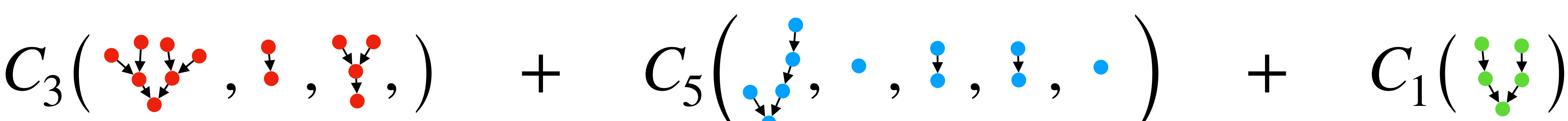
Just a finite set with a transition function (A, f) modulo isomorphism



General shape of a dynamical system

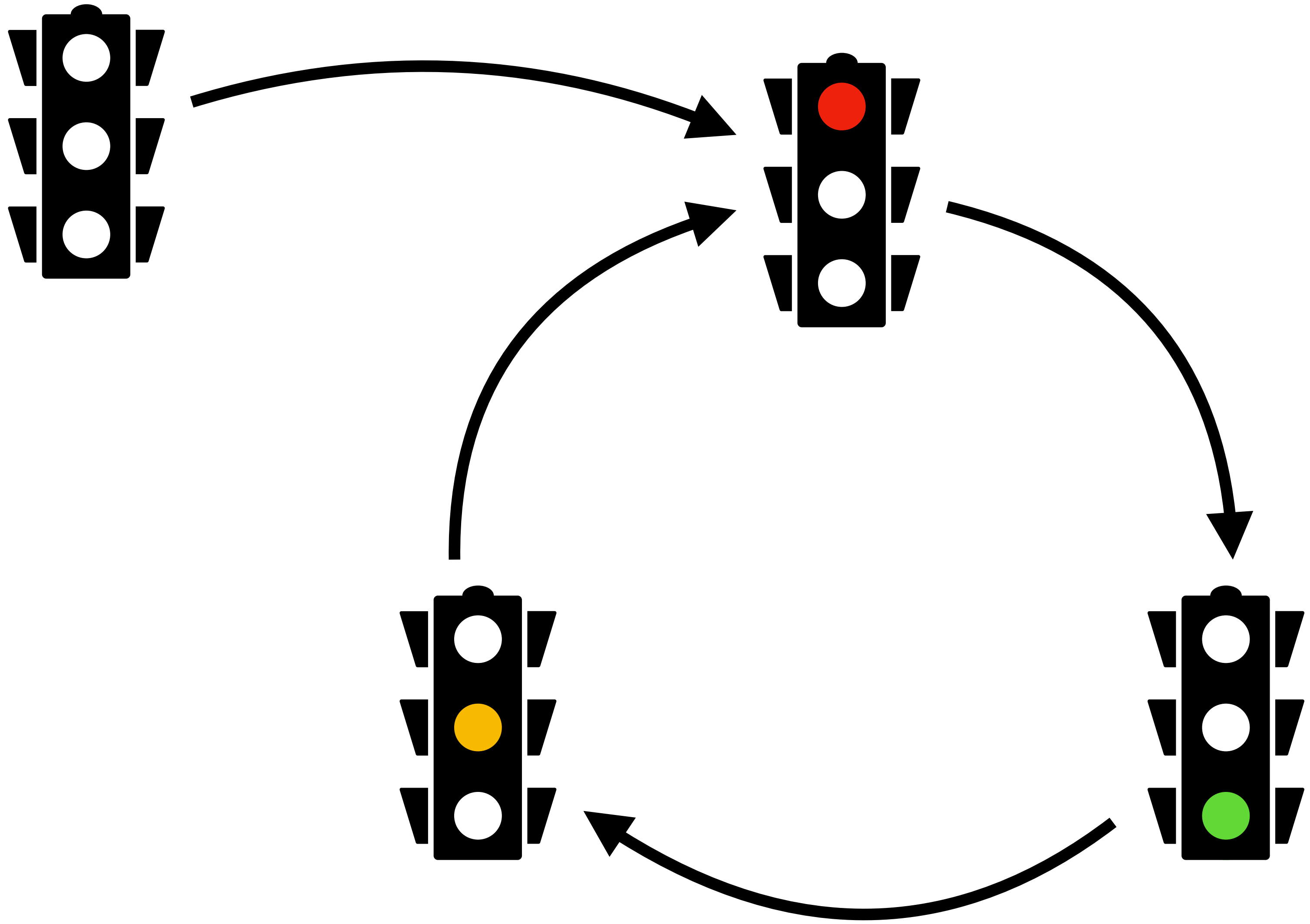
A few limit cycles with trees going in





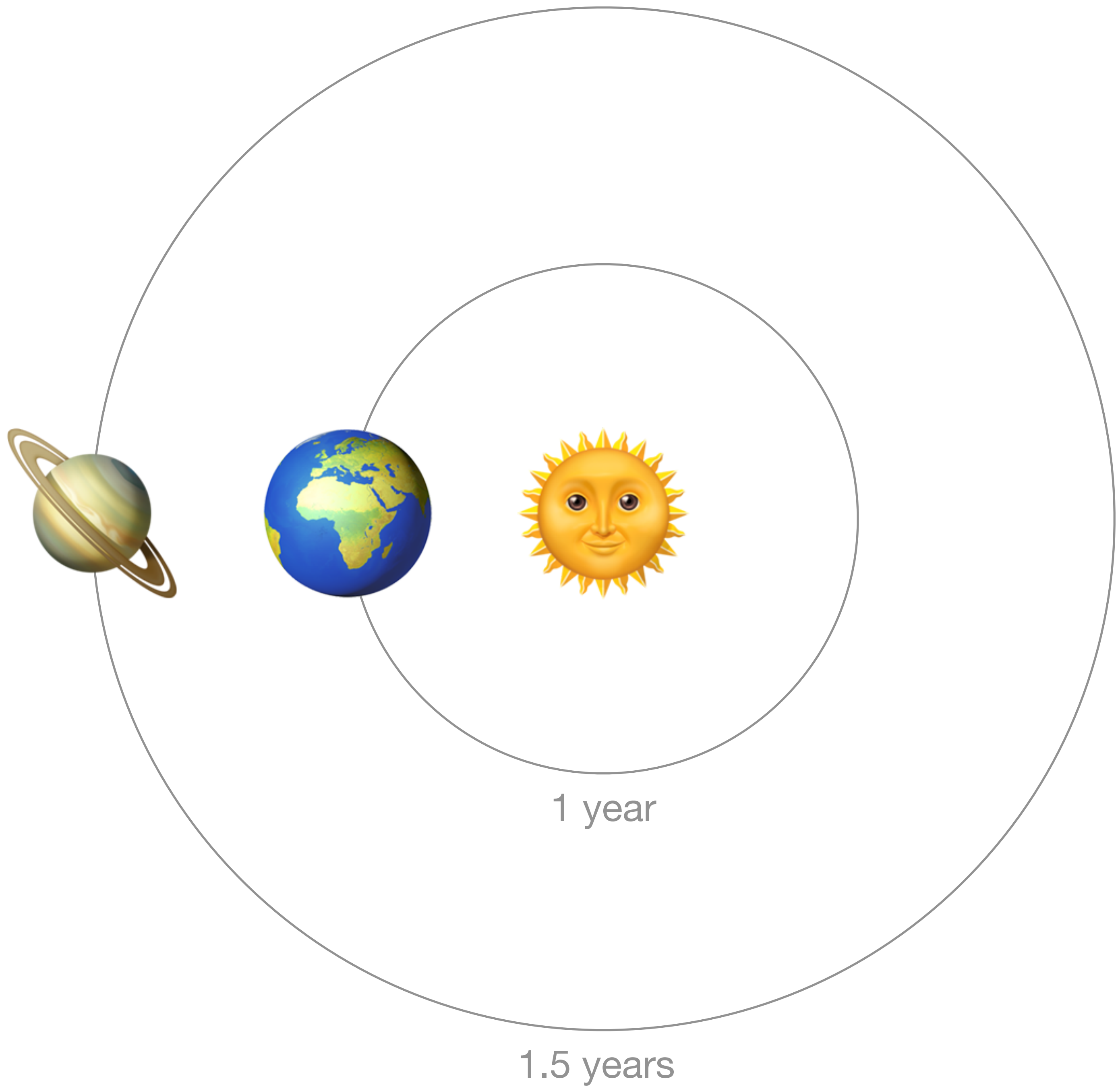
A toy example from engineering

Traffic lights

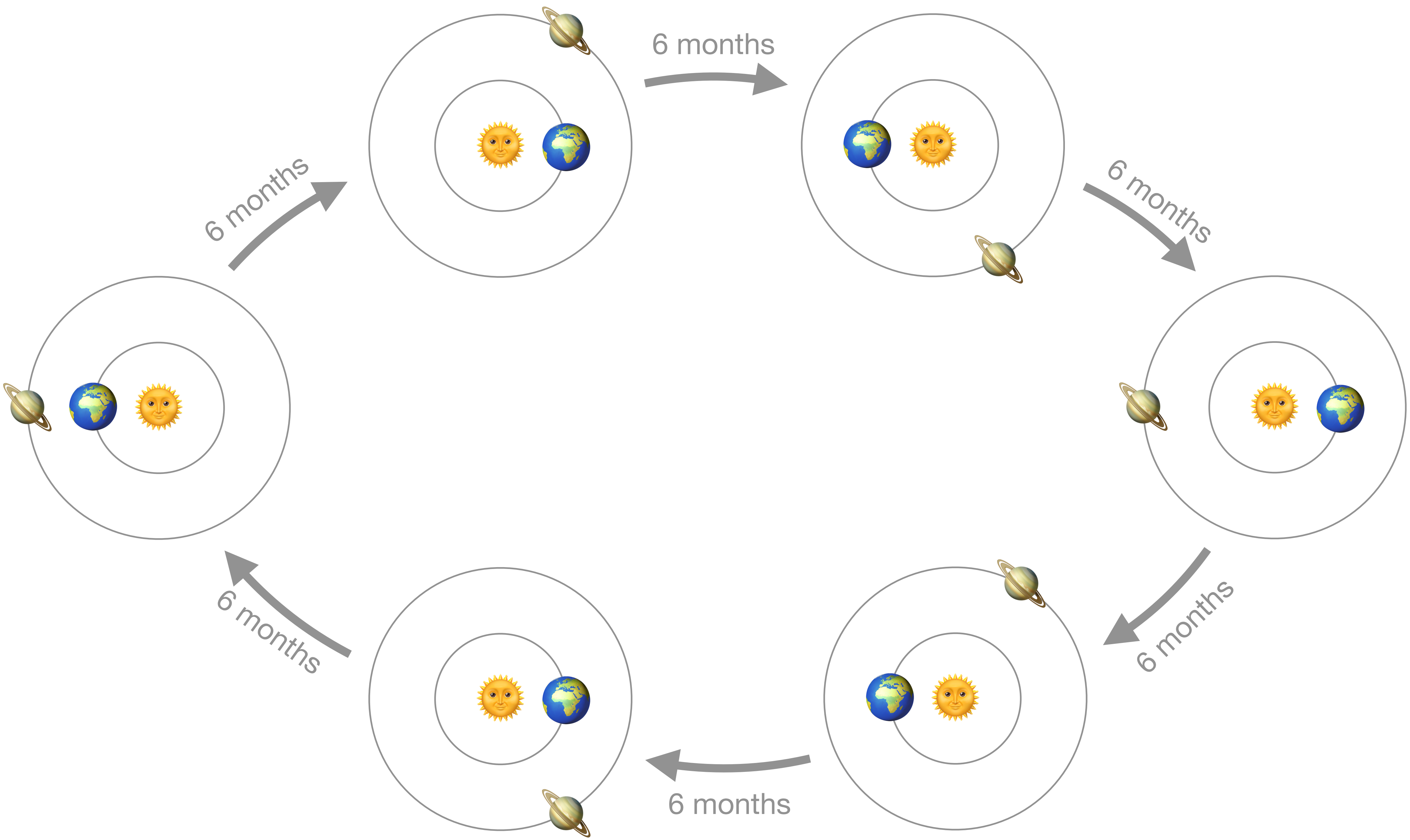


A toy example from science

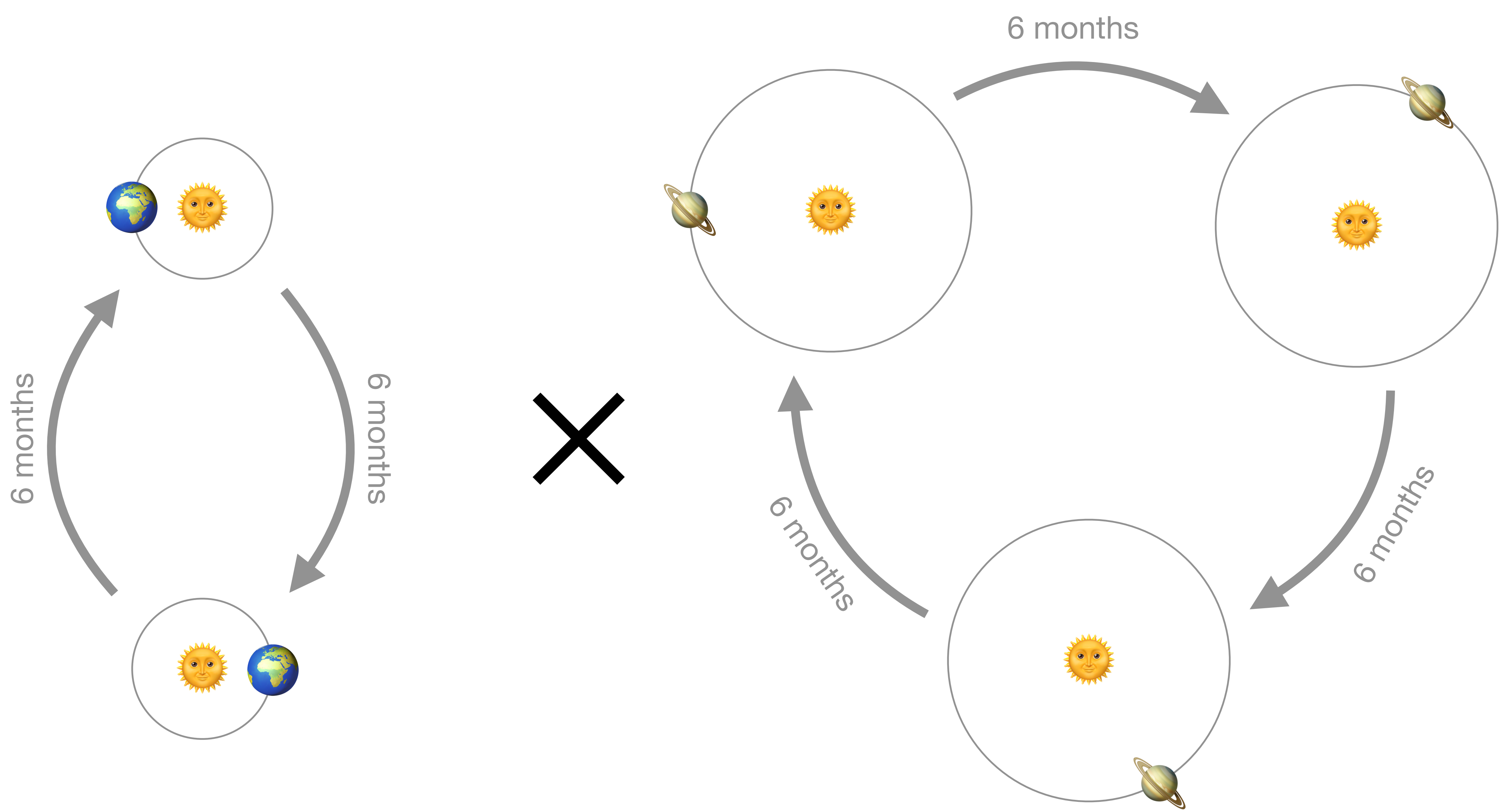
A planetary system



Evolution in time

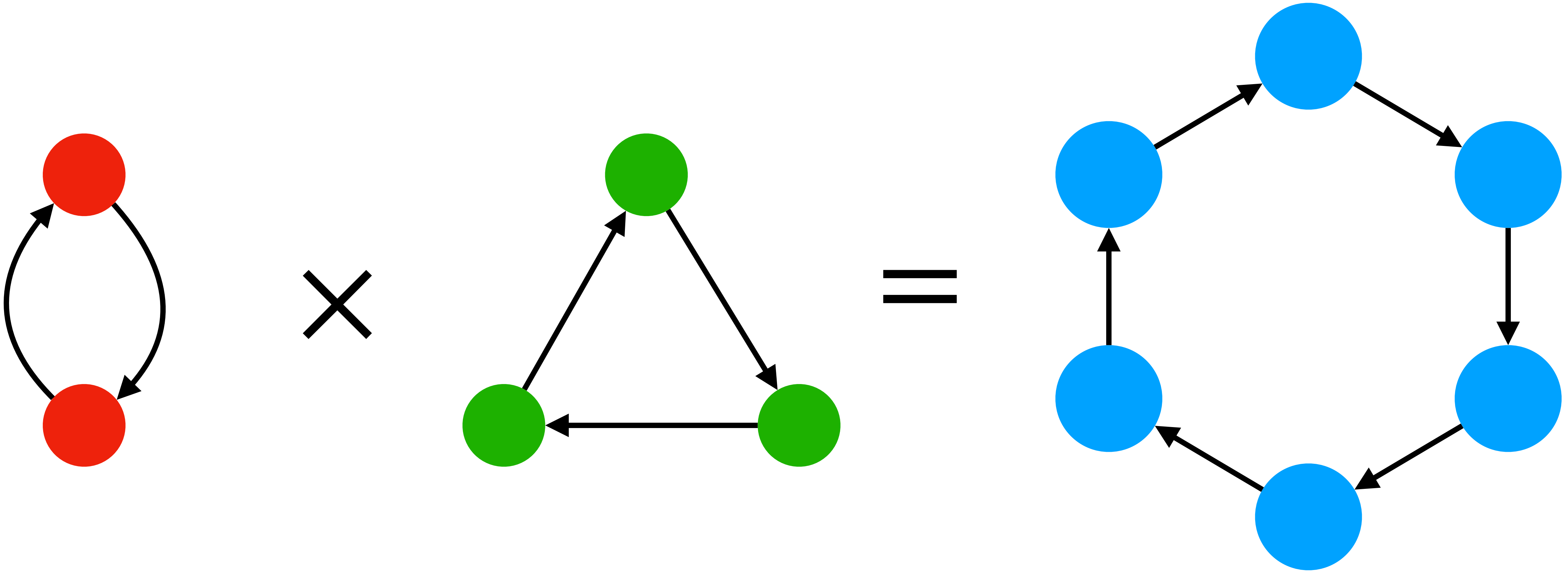


Decomposing the system



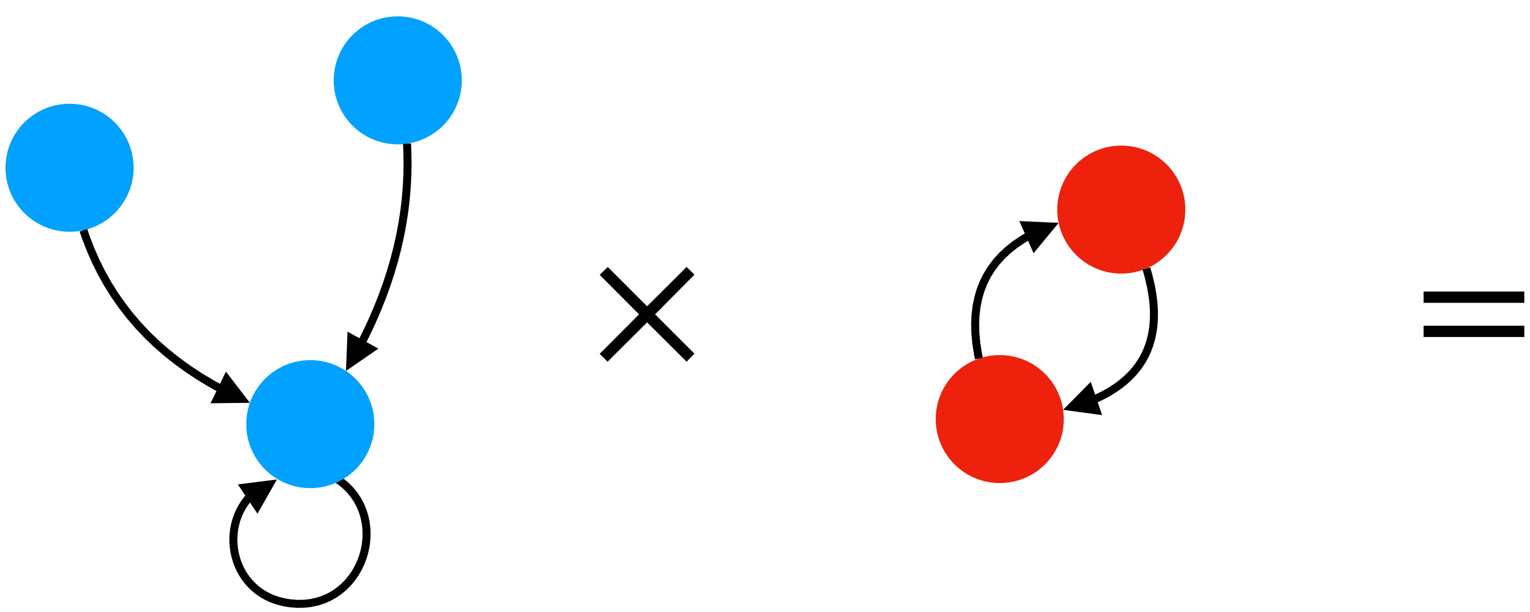
What if our instruments are less sophisticated?

Abstract evolution of the system

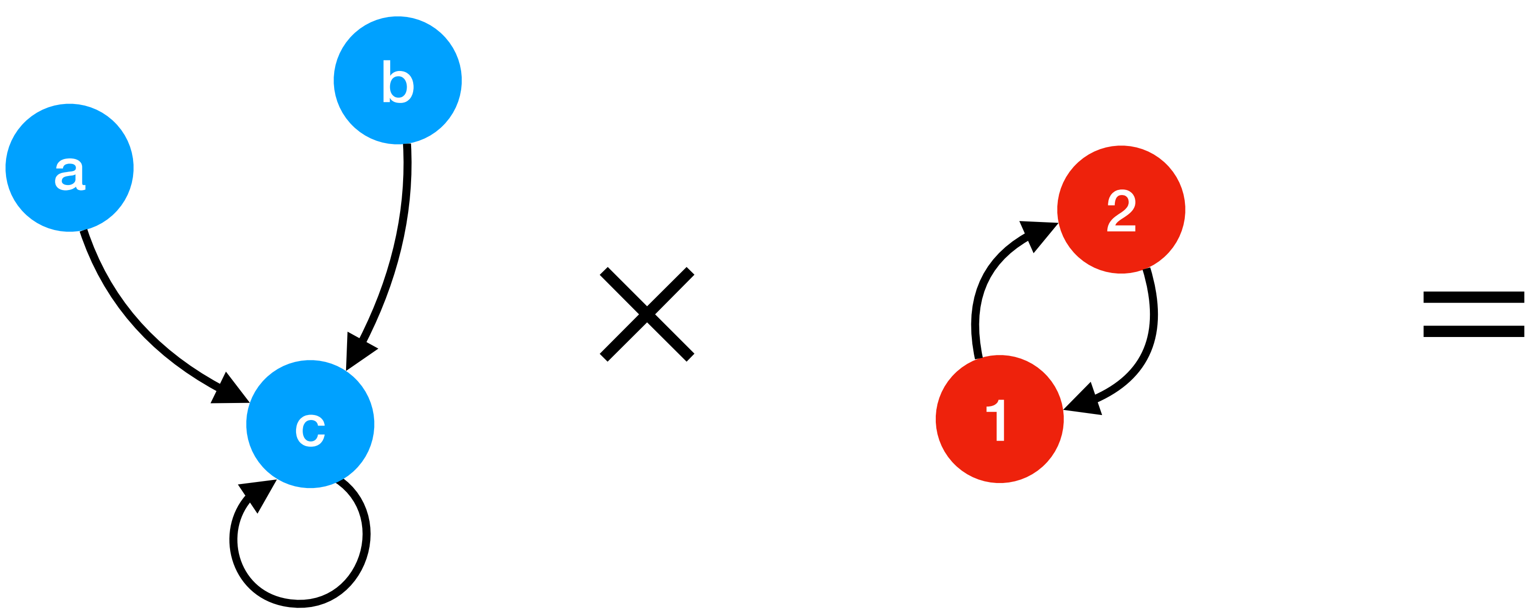


Product of dynamical systems

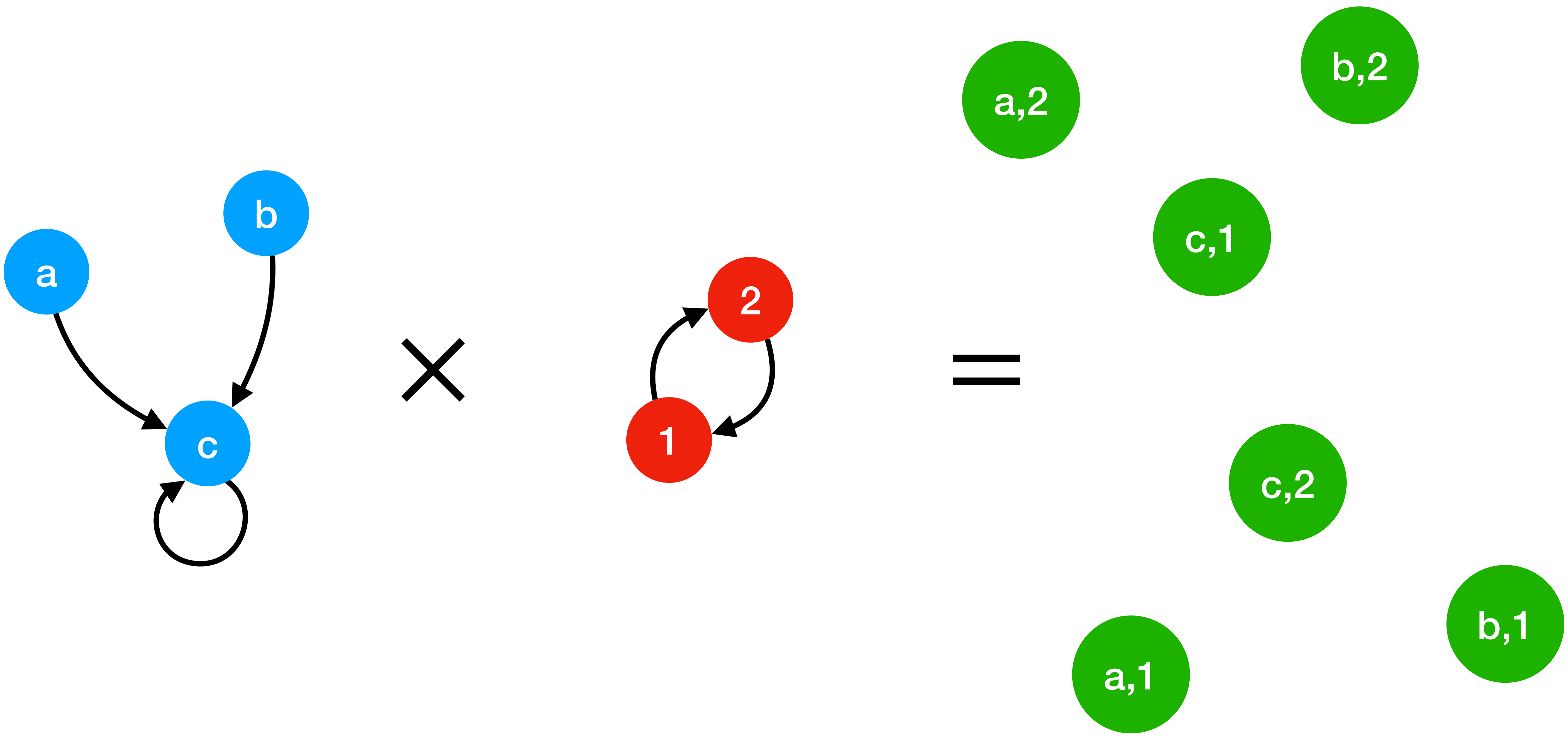
Product of systems



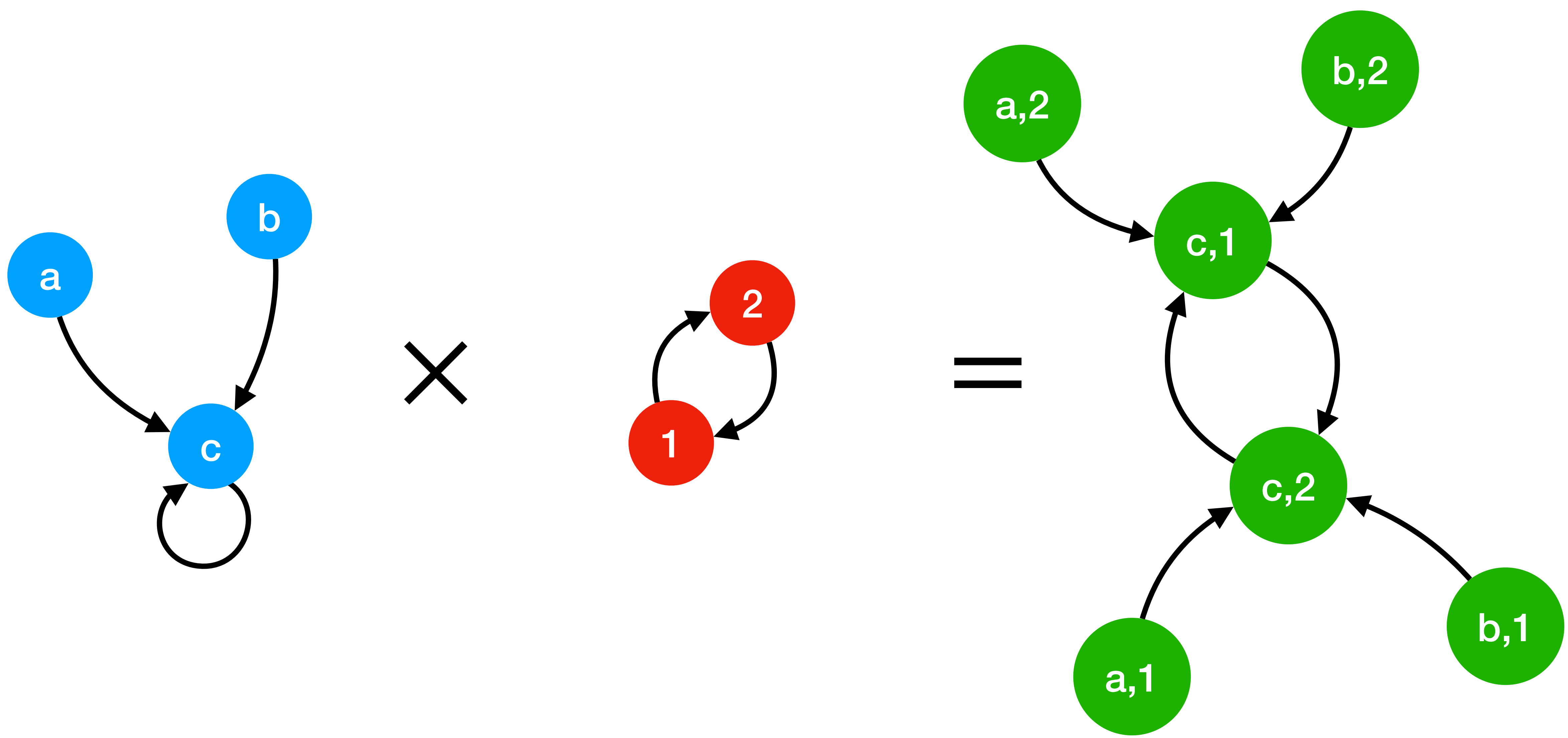
Give temporary names to the states



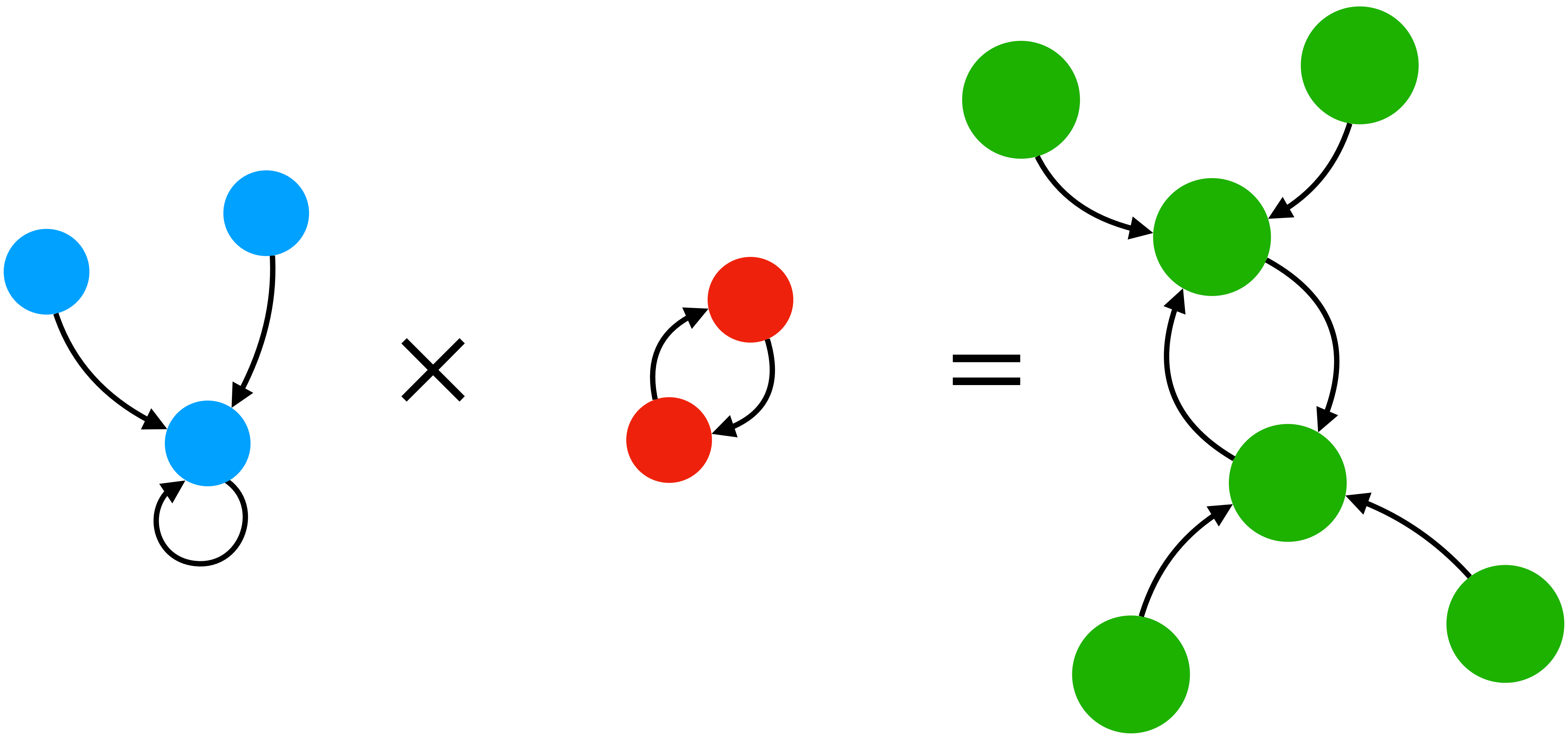
Compute the Cartesian product



Add the arcs between states

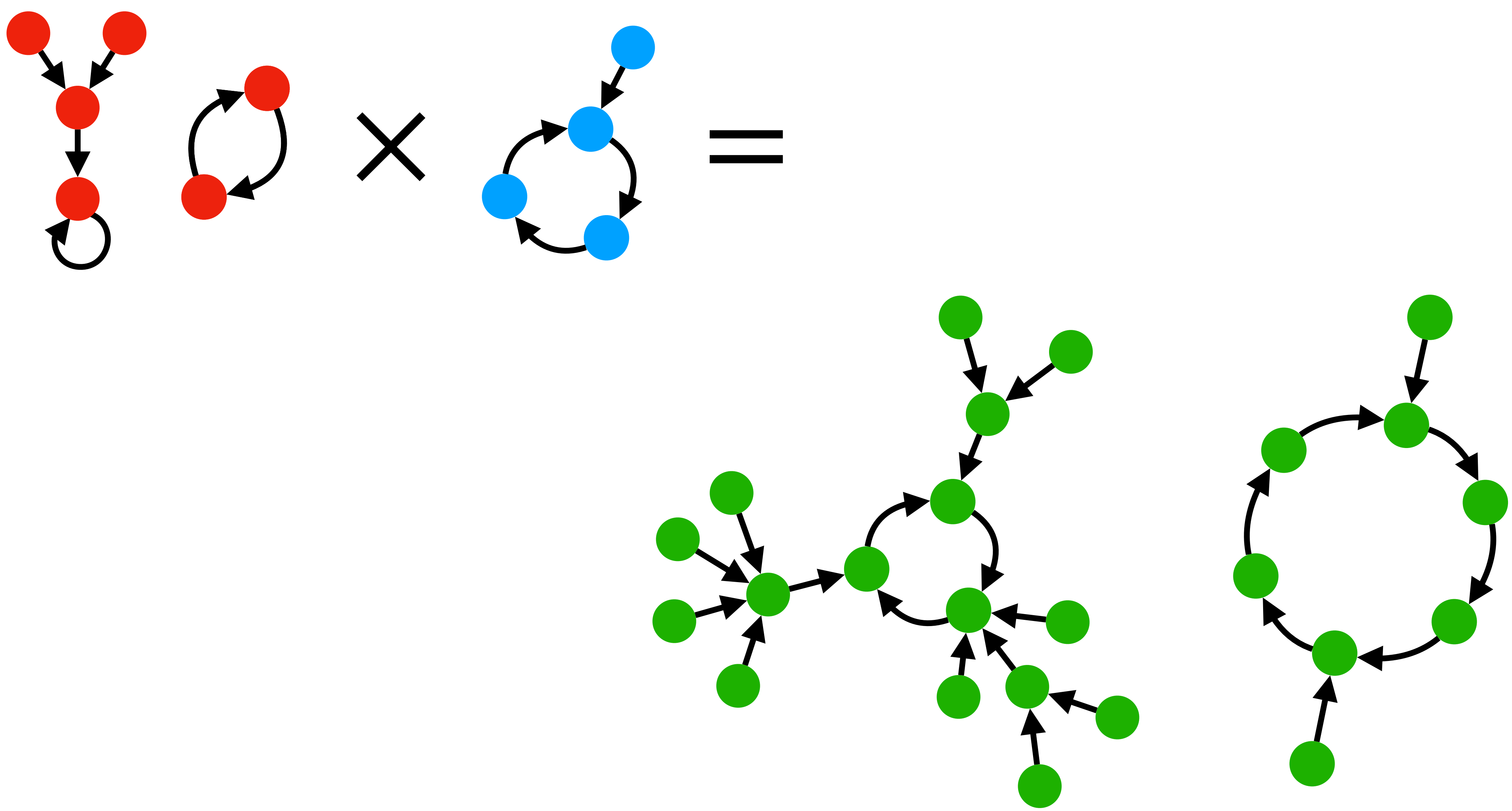


Forget the names once again



Products “preserve” behaviours

A **is a minor of** $A \times B$ **for** $B \neq \emptyset$



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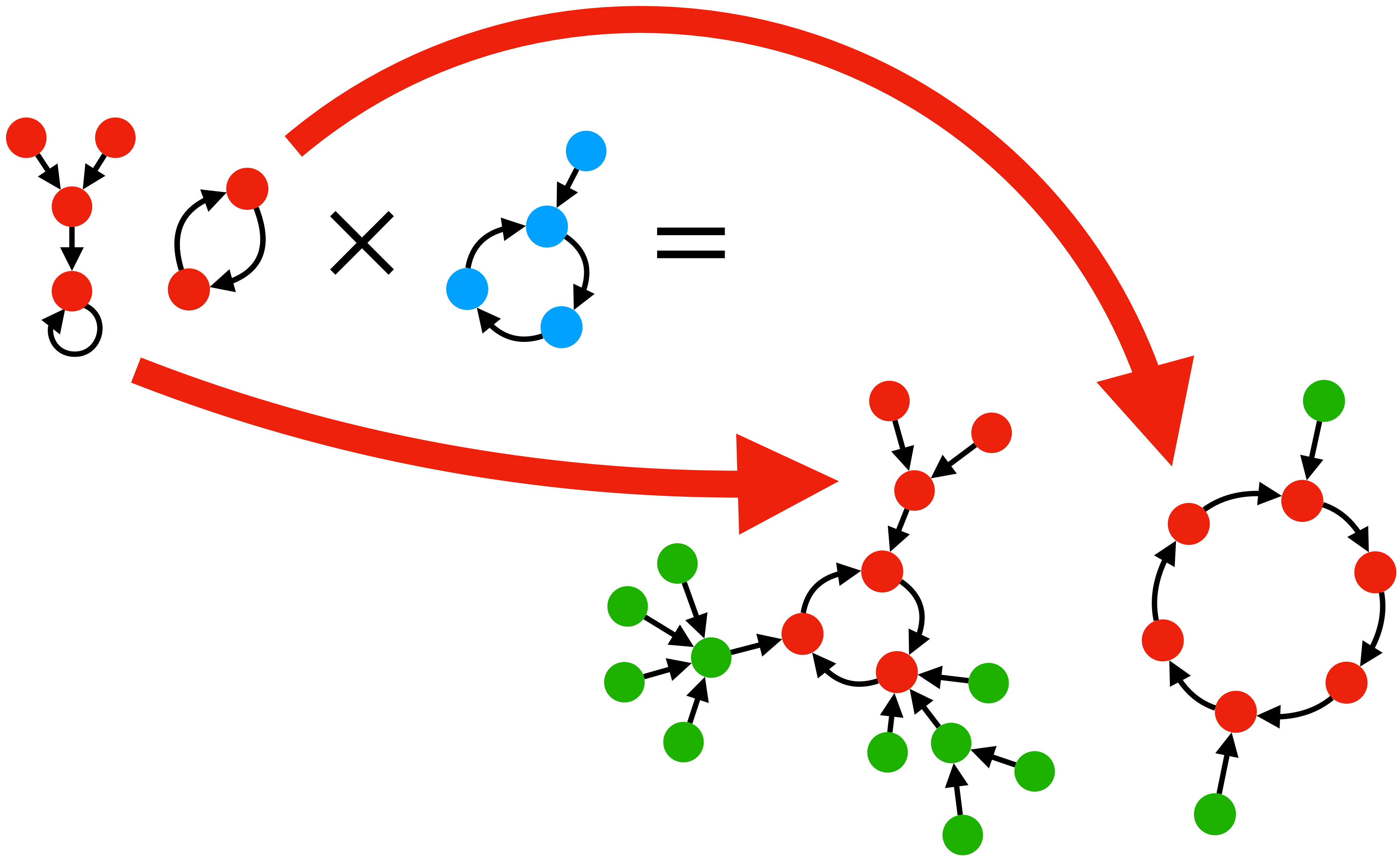

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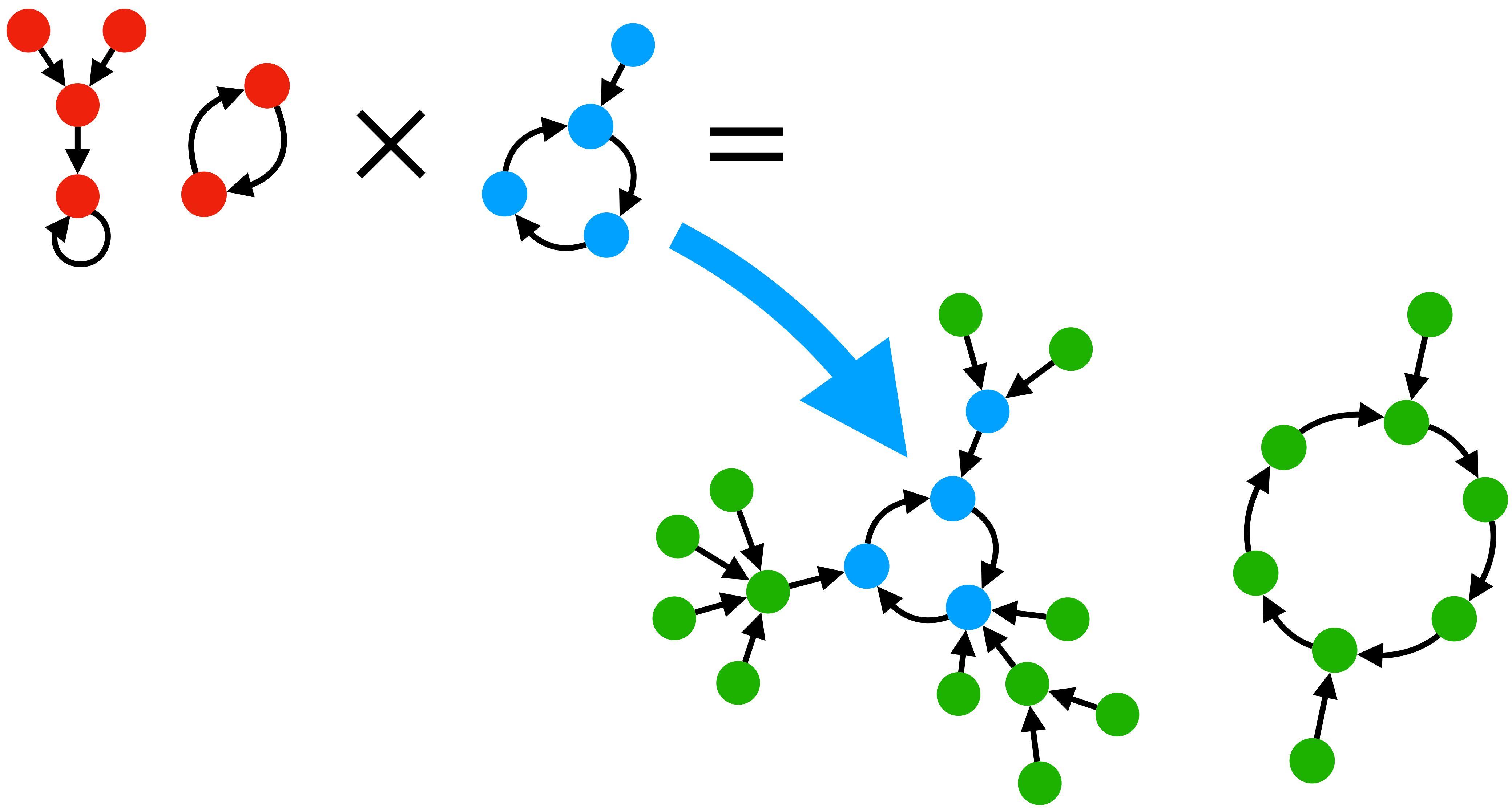

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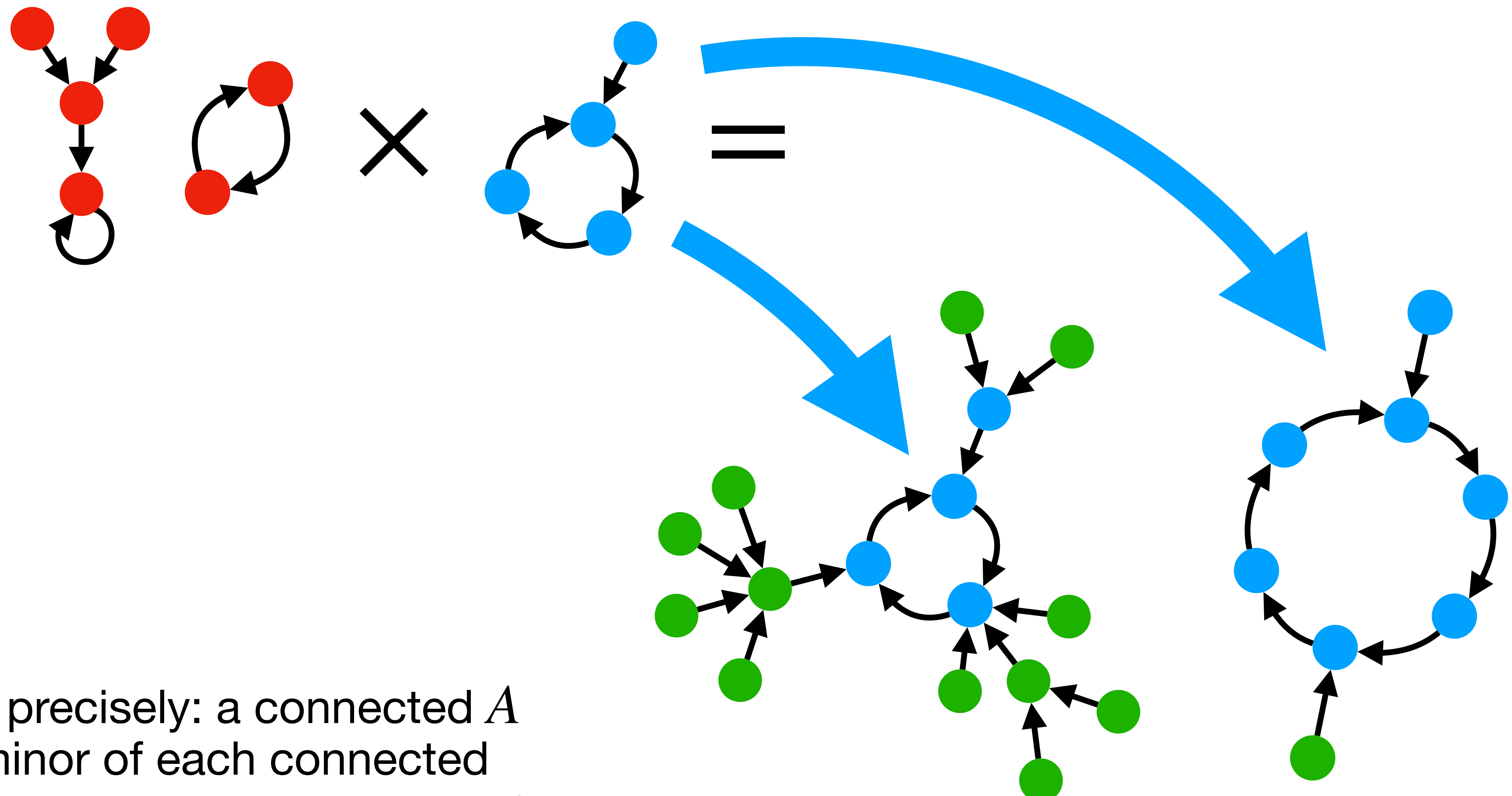
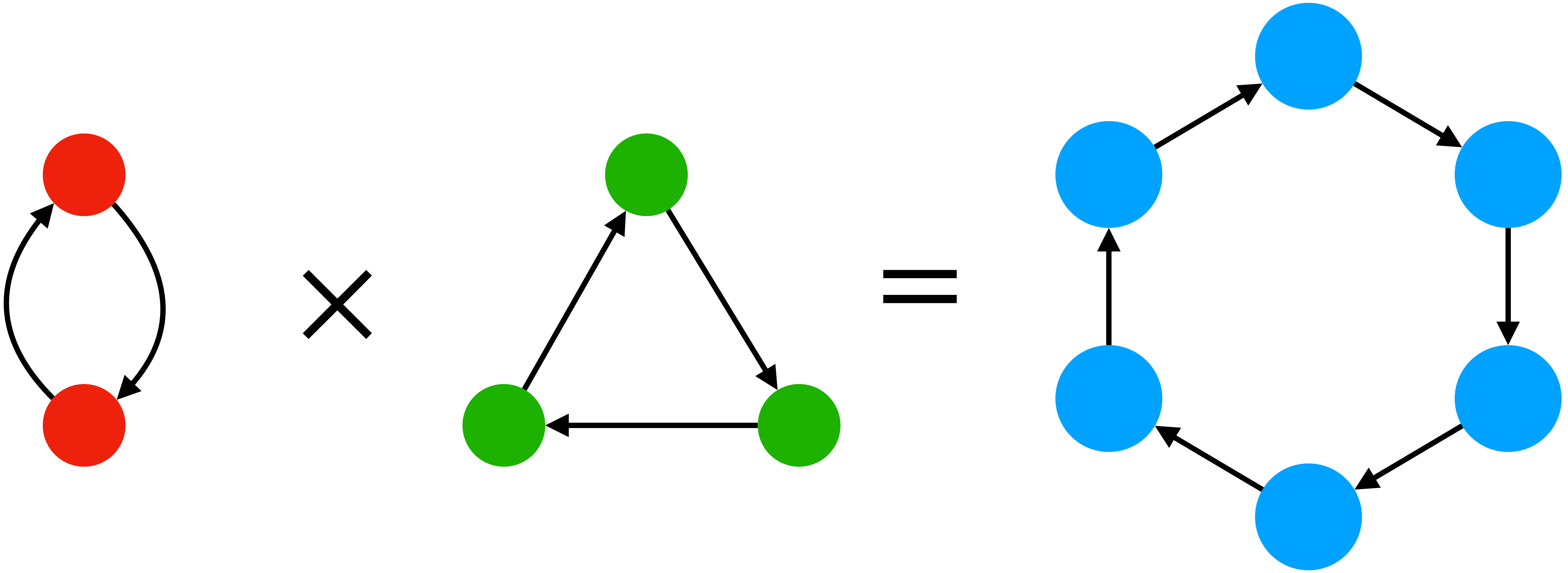


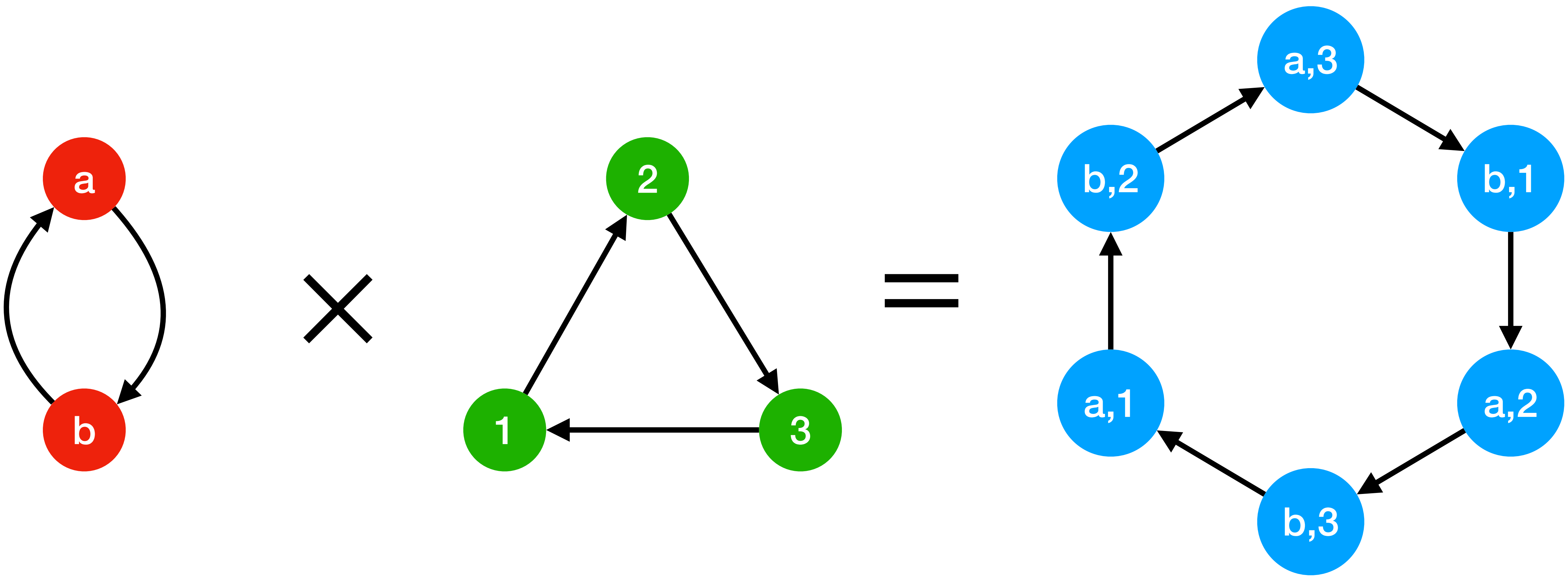
more precisely: a connected A
is a minor of each connected
component of $A \times B$ for $B \neq 0$

Back to our planetary system

Decomposition

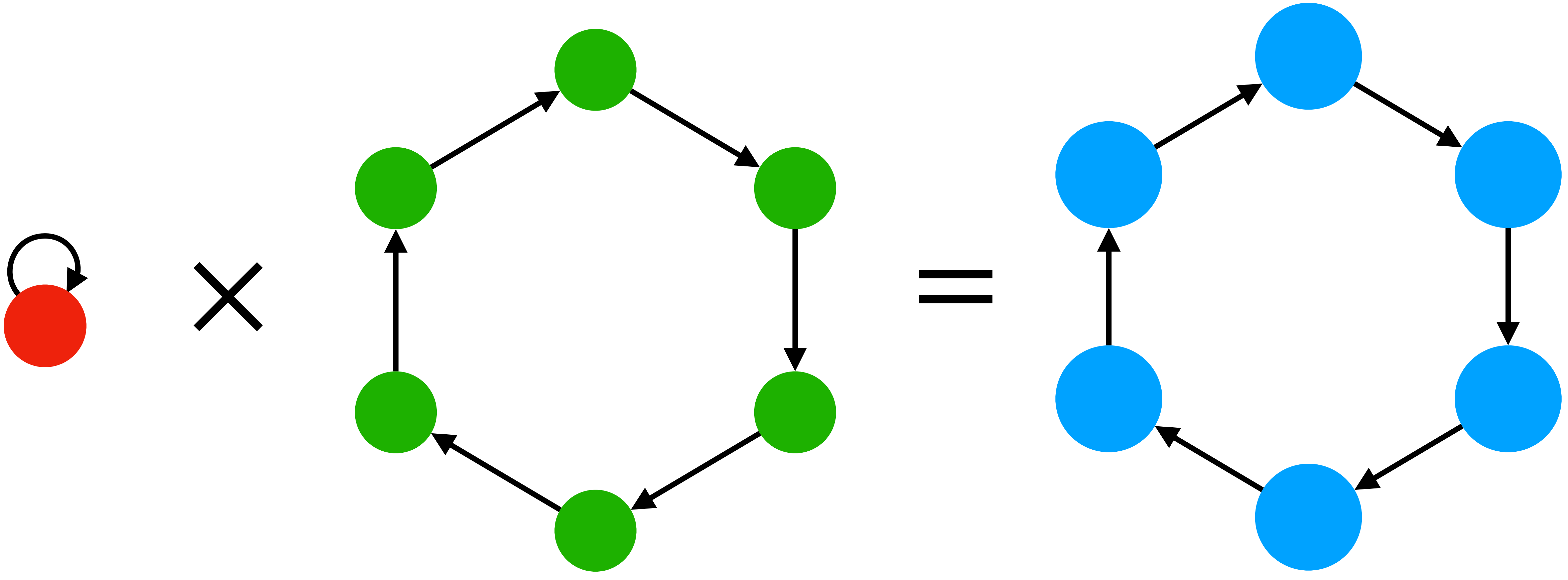


Decomposition

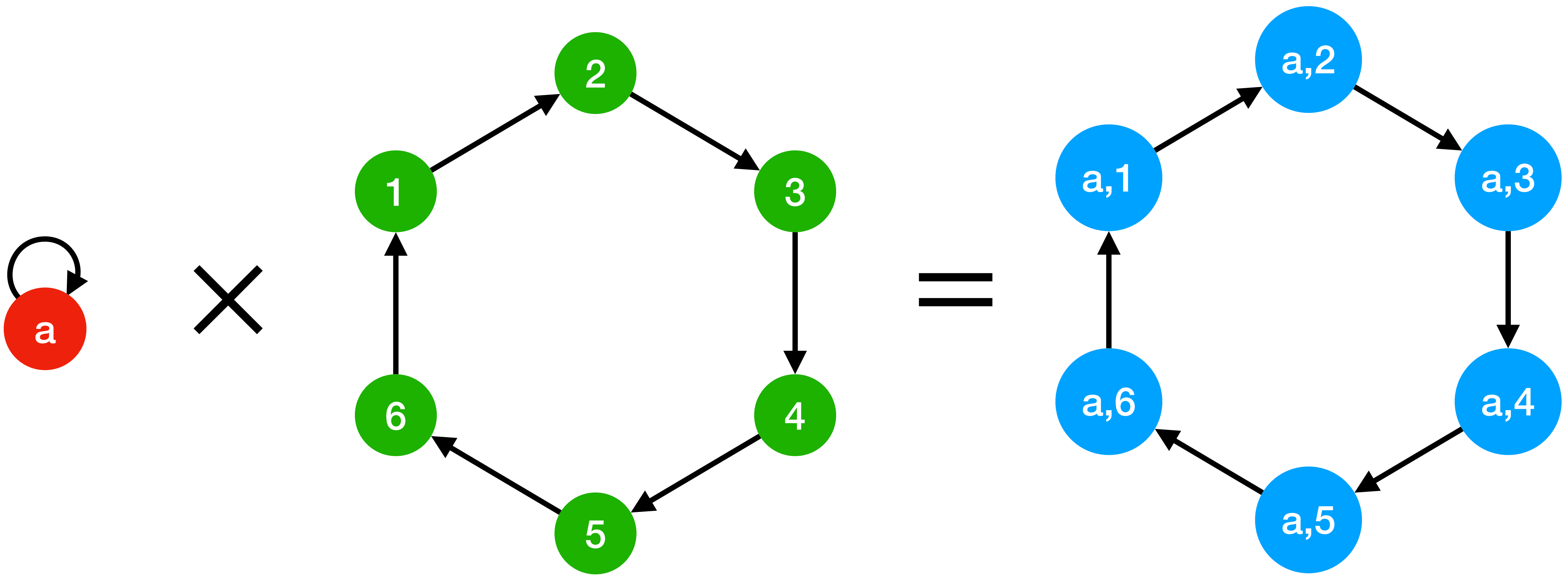


Any other decomposition?

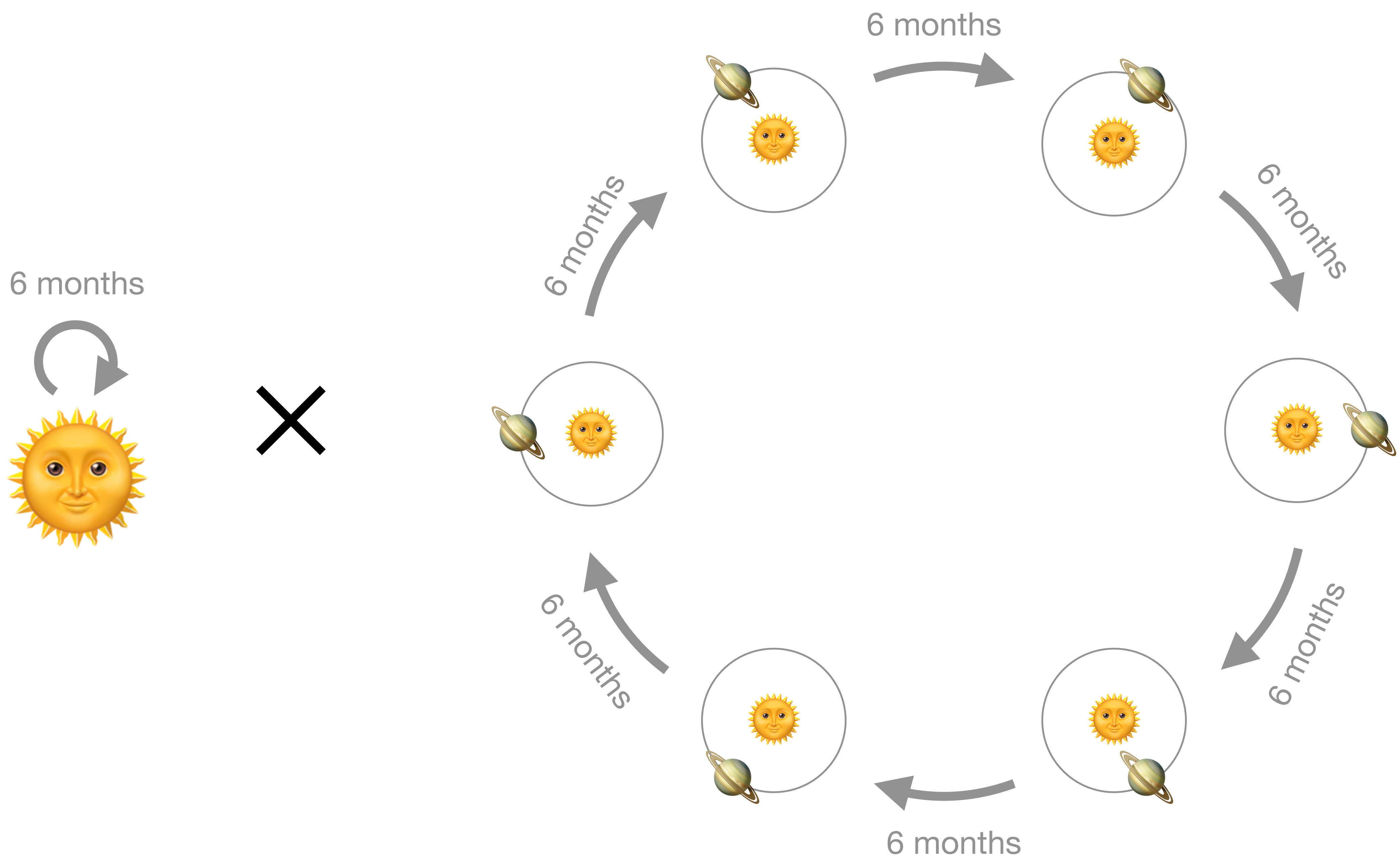
Another decomposition



Another decomposition

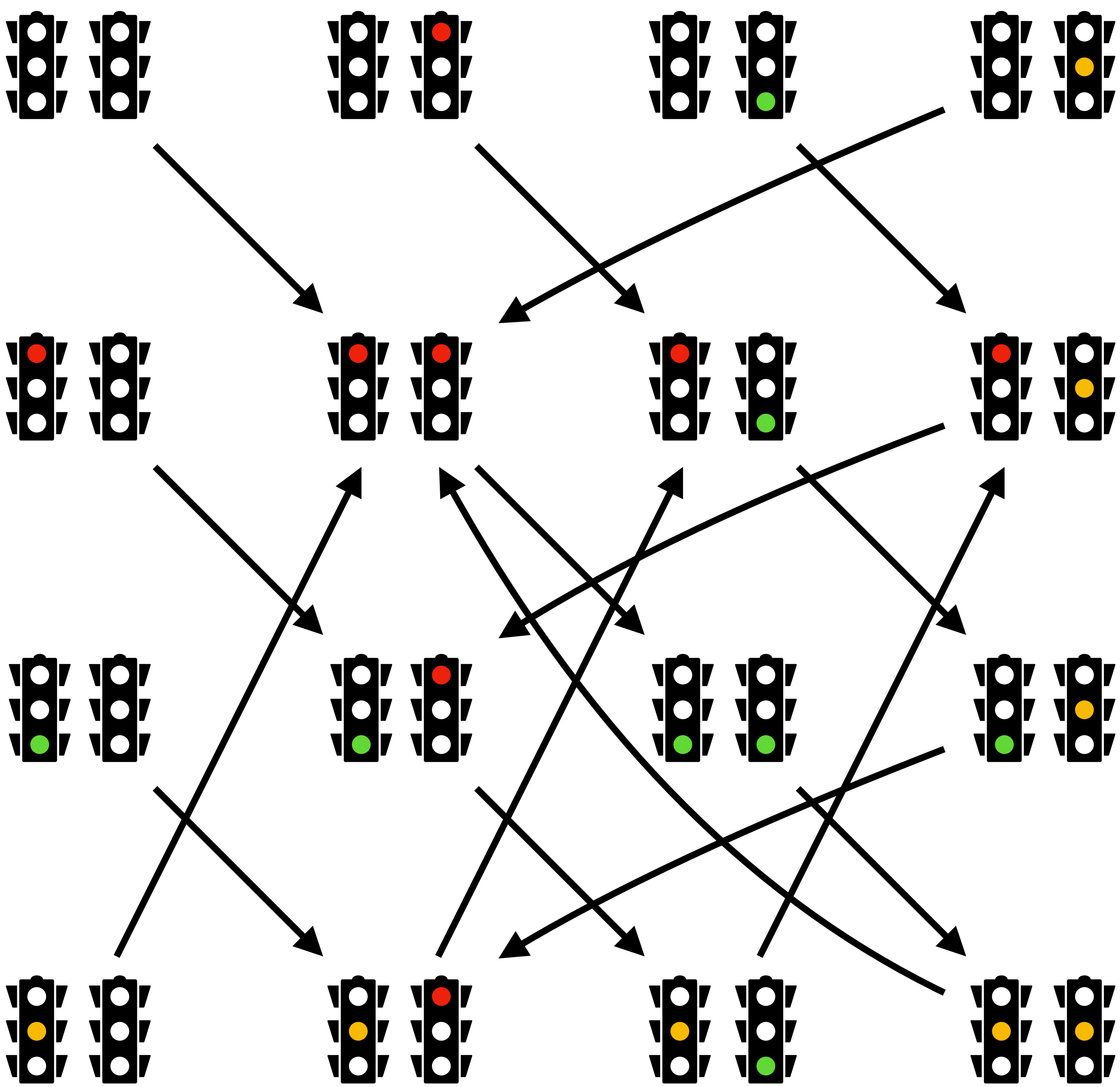


More concretely…

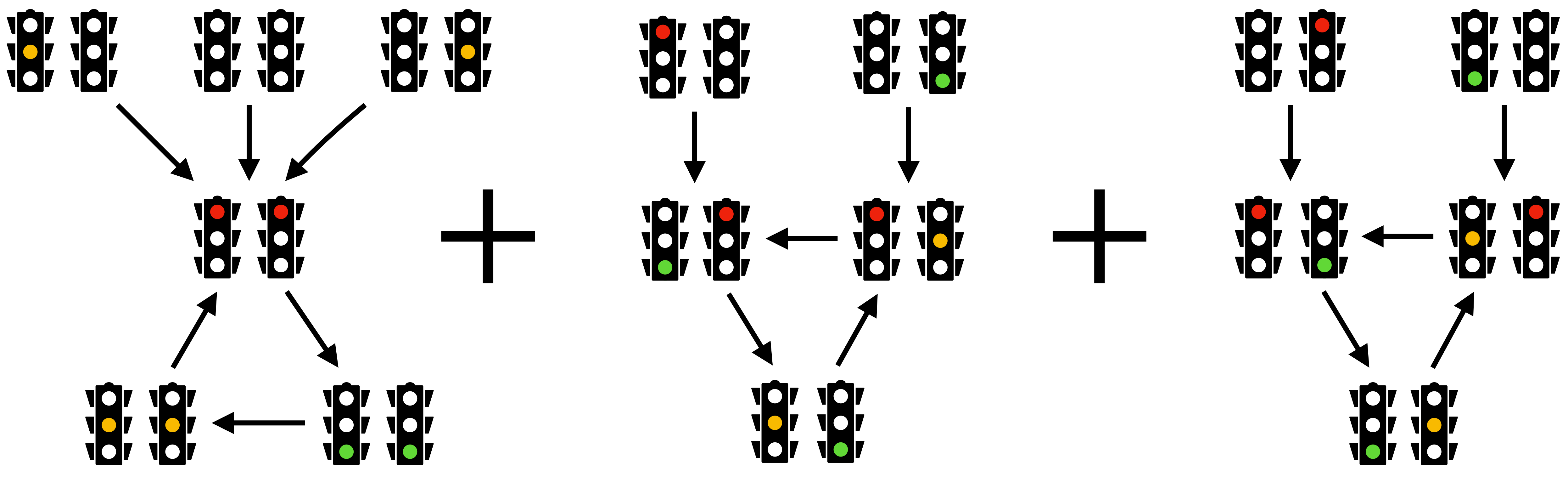


Untangling complex systems

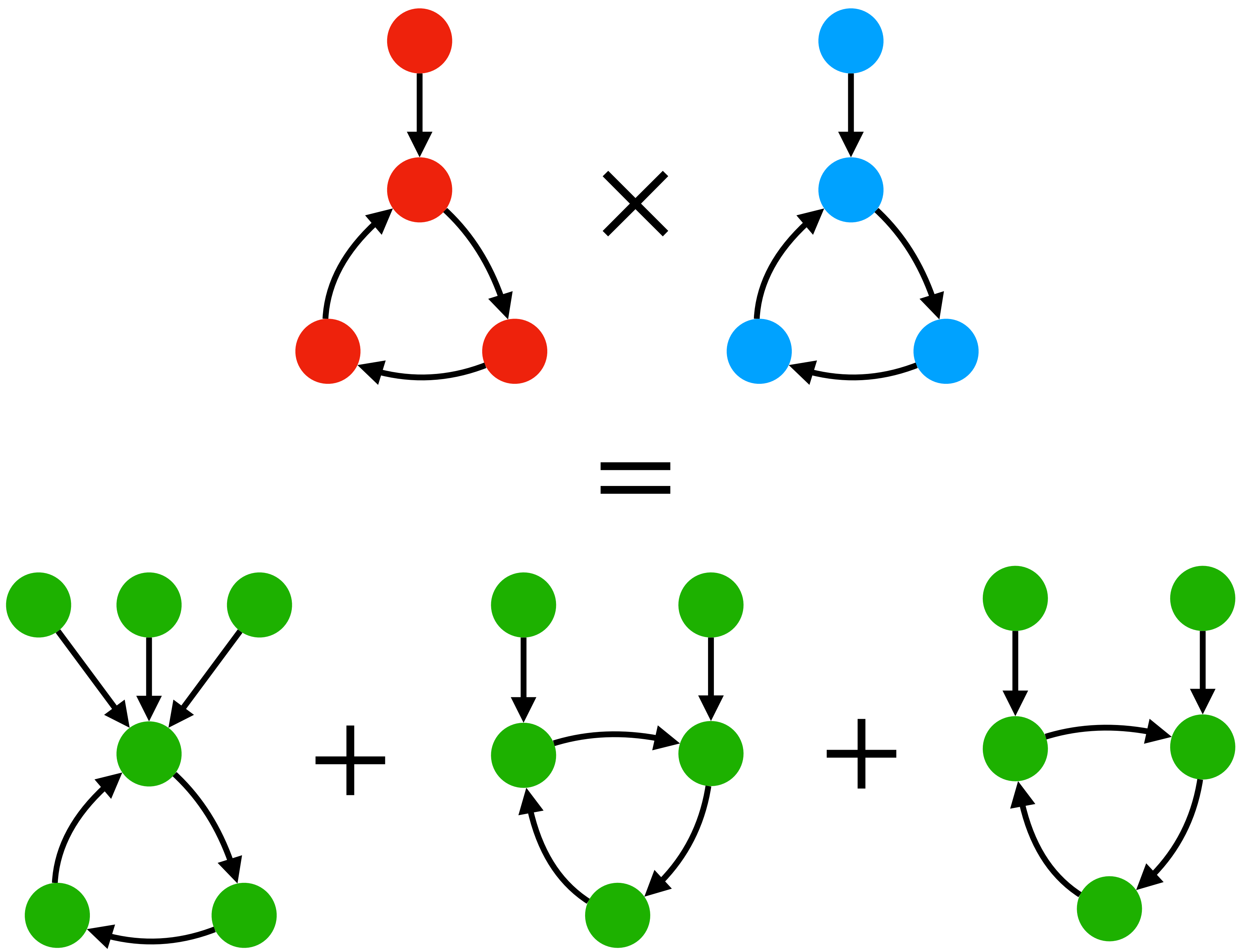
Traffic lights at a crossroads



Traffic lights at a crossroads



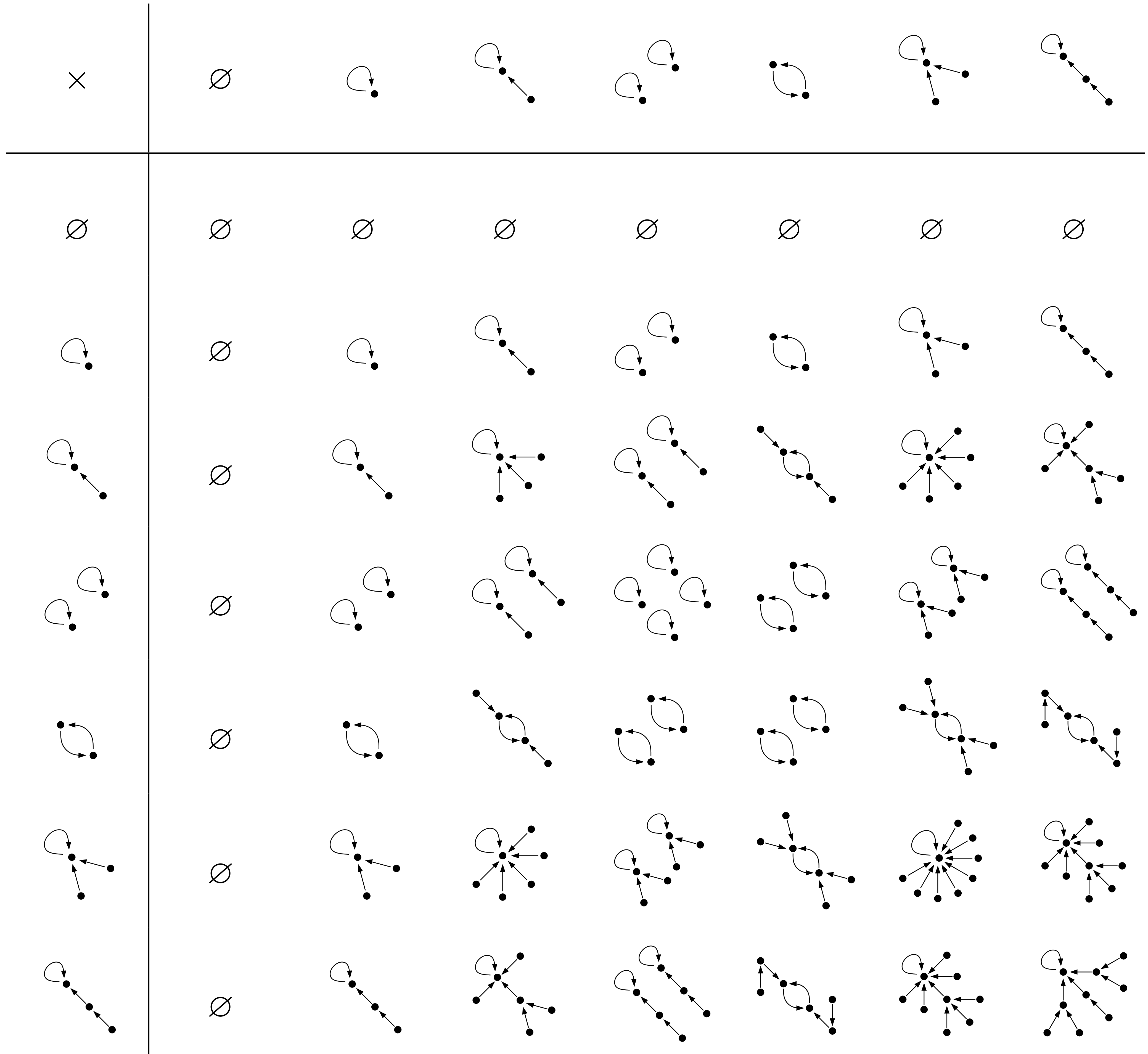
More abstractly…



The operations $+$ and $\times$ are a commutative semiring

- Commutative: $X + Y = Y + X$ and $X \times Y = Y \times X$
- Associative: $X + (Y + Z) = (Y + X) + Z$ and $X \times (Y \times Z) = (Y \times X) \times Z$
- Neutral elements: $\varnothing + X = X$ and  $\times X = X$
- Distributive: $X \times (Y + Z) = X \times Y + X \times Z$
- Multiplication by zero: $\varnothing \times X = \varnothing$

Multiplication table



Reducibility of dynamical systems

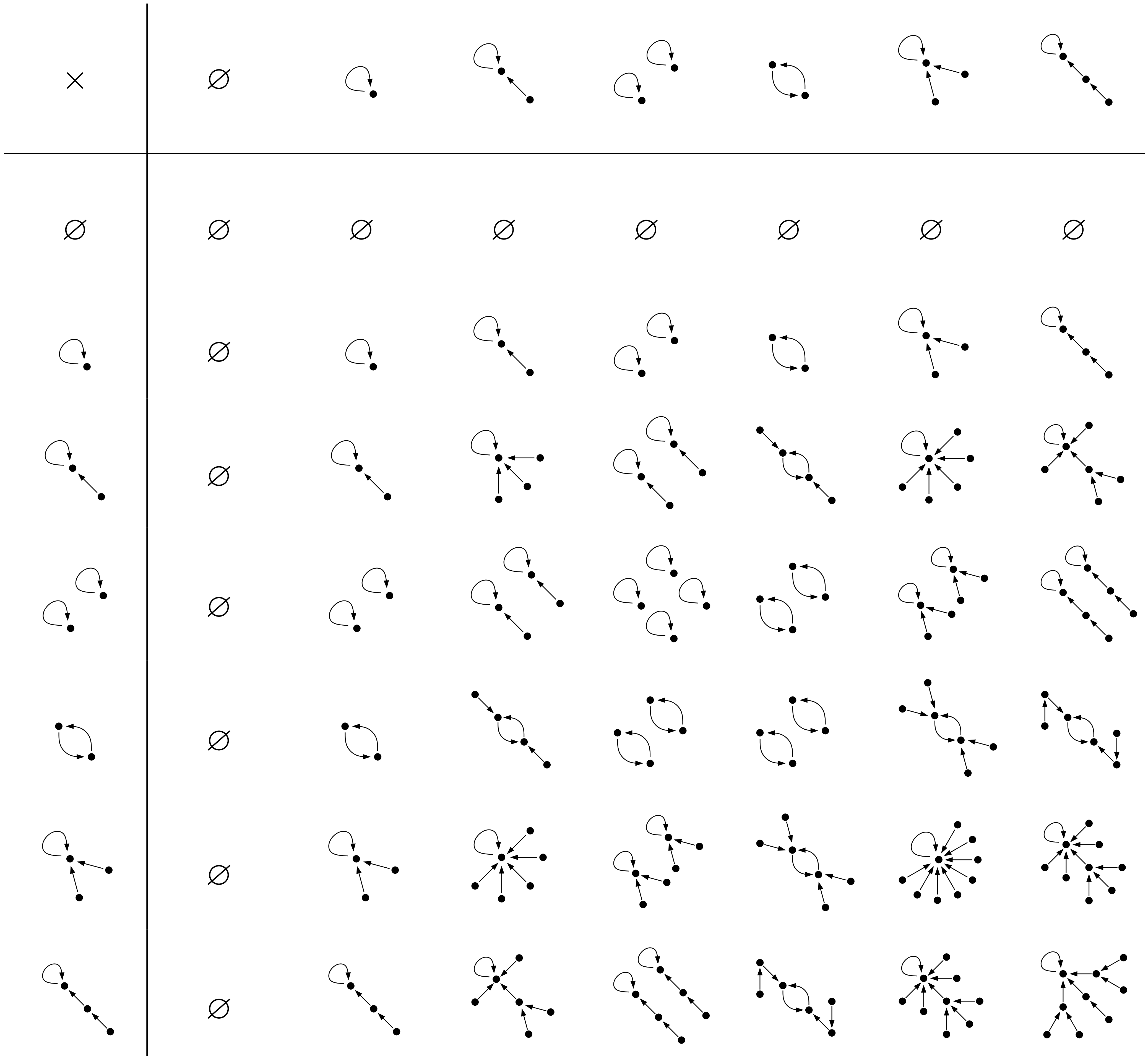
Most dynamical systems are irreducible

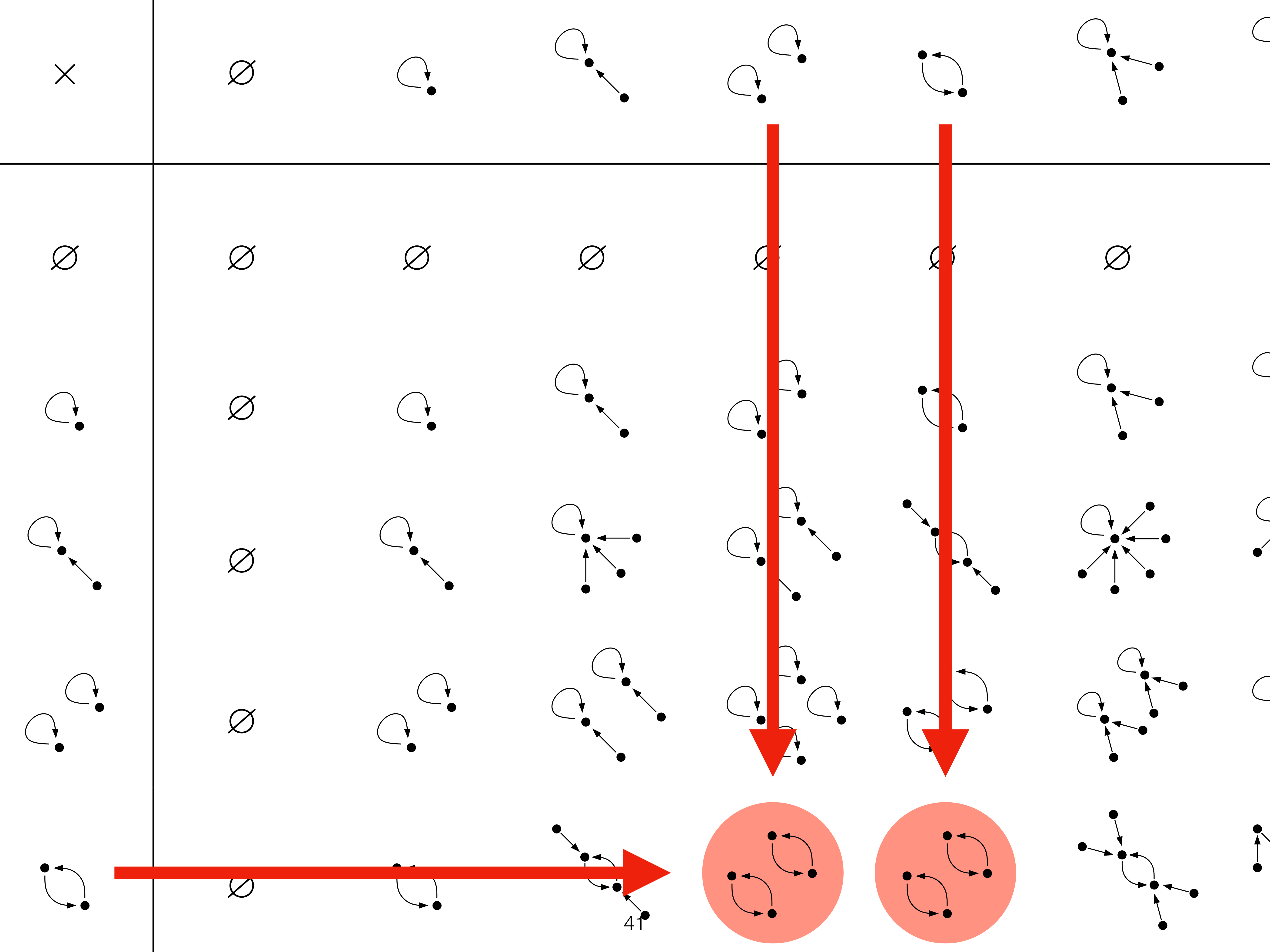
- Formally:

$$\lim_{n \to \infty} \frac{\text{number of reducible systems over} \leq n \text{ states}}{\text{total number of systems over} \leq n \text{ states}} = 0$$

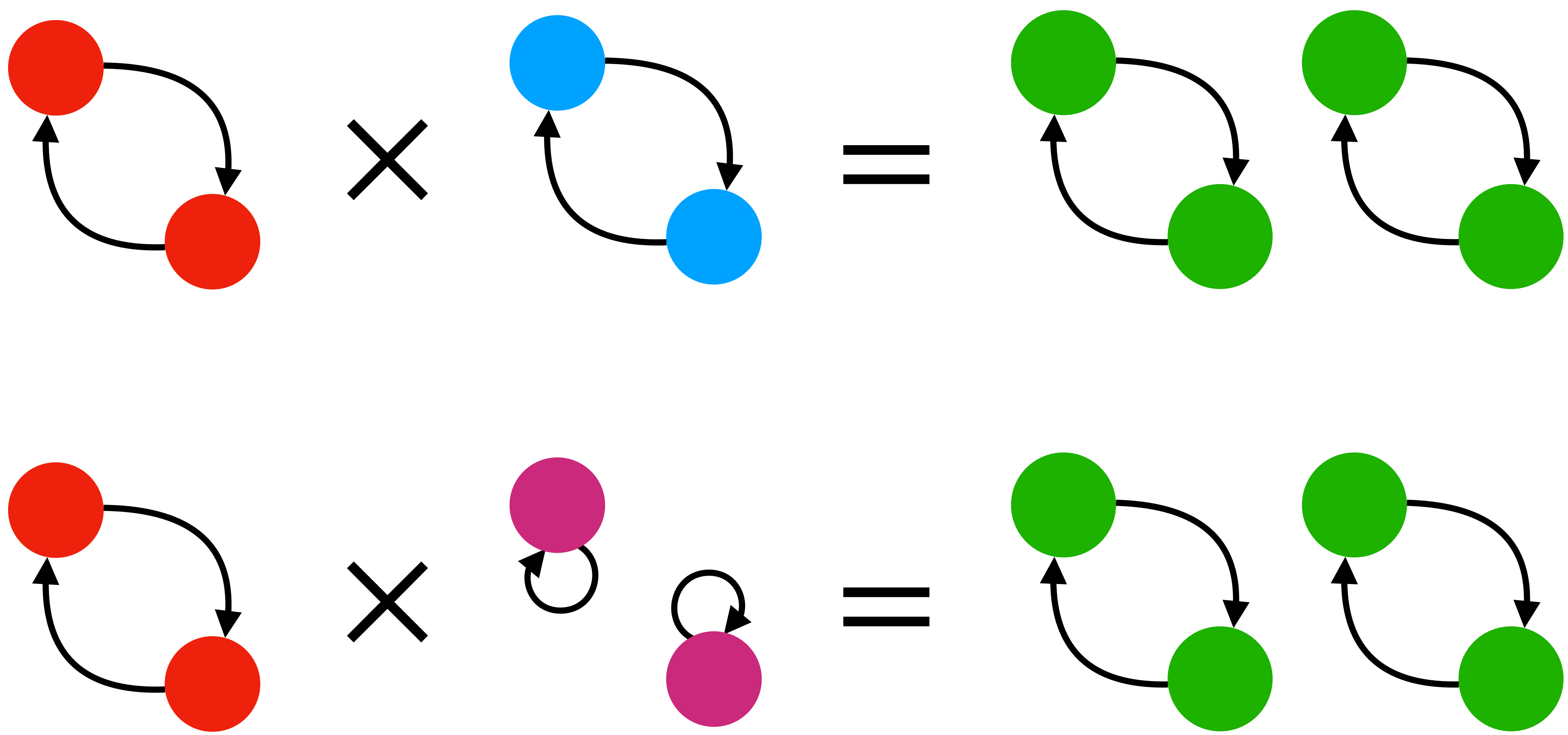
- Notice that this is the opposite of $\mathbb{N}$, where irreducible (aka prime) integers are scarce

No unique factorisation into irreducibles!

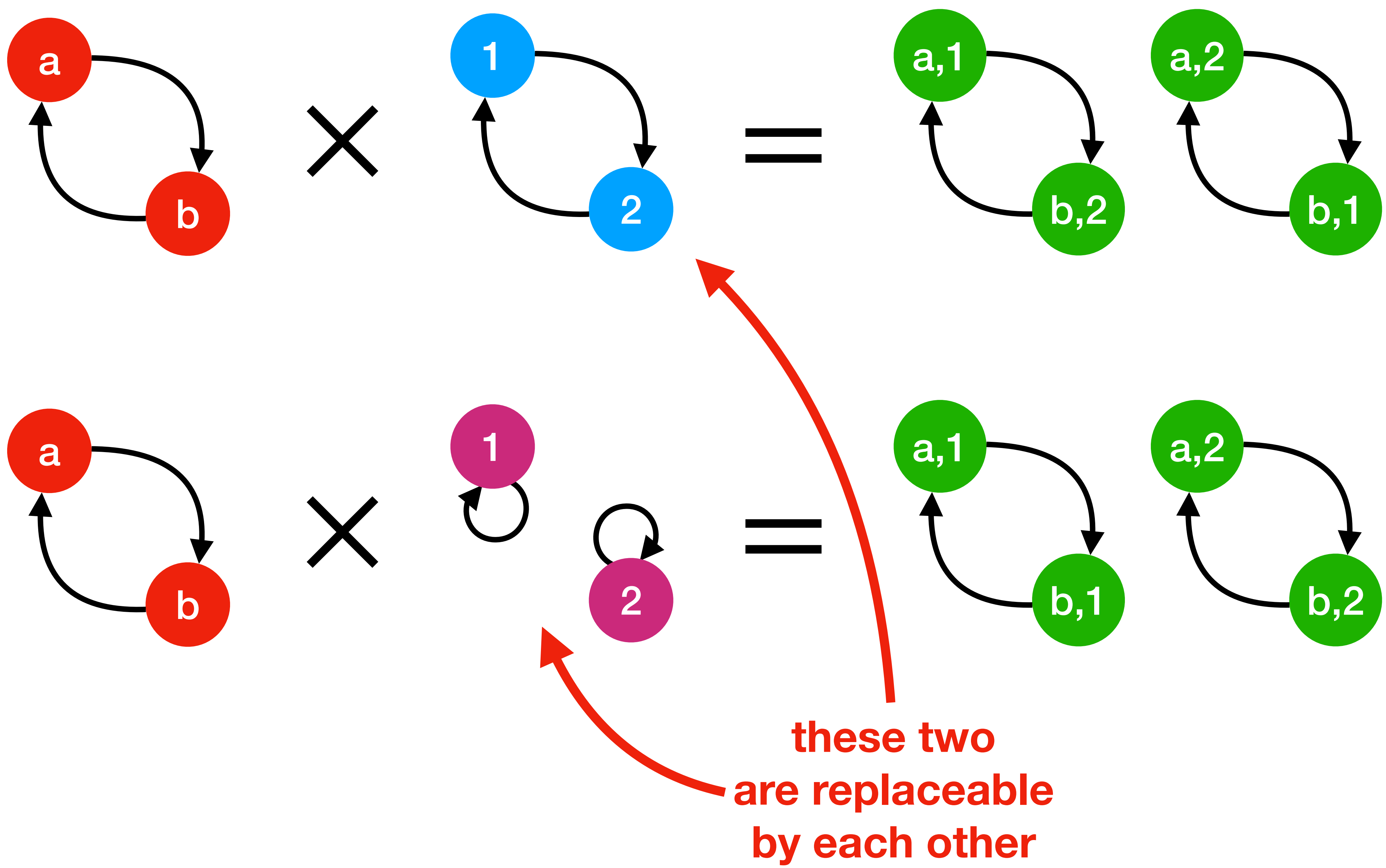




Multiple factorisations



Multiple factorisations



Prime systems

- A prime is a system P such that, whenever it appears in a factorisation into irreducibles of $A \times B$, it appears in the factorisation of either A or B
- In other words, if P divides $A \times B$ then it divides A or B
- If a prime appears in one factorisation of a system, then it appears in all the others as well (it is irreplaceable)

Do primes exist?

- After a lot of headaches (including at least two master's internships)…

- …it turns out that primes do not exist!

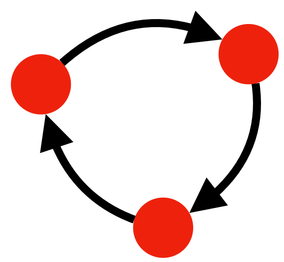
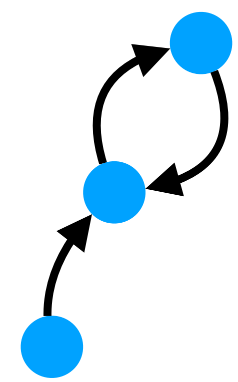
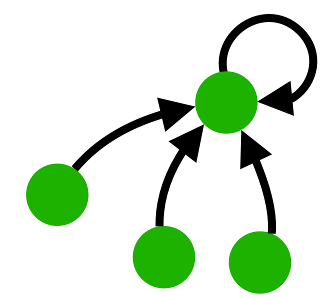
- The problem was solved under a different name in the 60s in the context of universal algebra

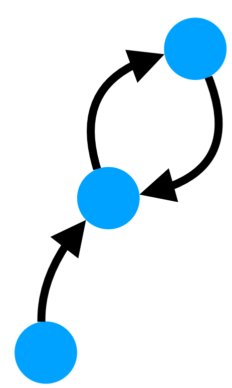
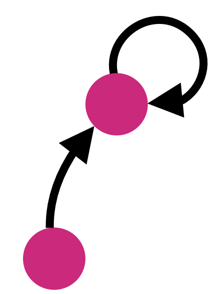
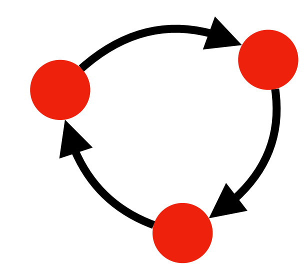
    - R. Seifert, On prime binary relational structures, Fundamenta Mathematicae (70)2, 1971

- Thanks to Barbora Hudcová (EPFL, Lausanne) for this discovery and Maximilien Gadouleau (Durham Uni) for sharing it

Equations for decomposing systems

Eqns over dynamical systems

 $X + Y^2 =$  $Z +$ 

$X =$  $Y =$  $Z =$ 

And now more details on solving equations…

Problèmes prouvés difficiles

- Théorème Les équations de la forme $P_1(\overrightarrow{X}) = P_2(\overrightarrow{X})$ avec P_1, P_2 deux polynômes sont indécidables.

- Théorème Soit $P(\overrightarrow{X}) = B$ une équation où le nombre de variables dans $\overrightarrow{X}$ est non borné. Décider si l'équation admet une solution est **NP**-complet.

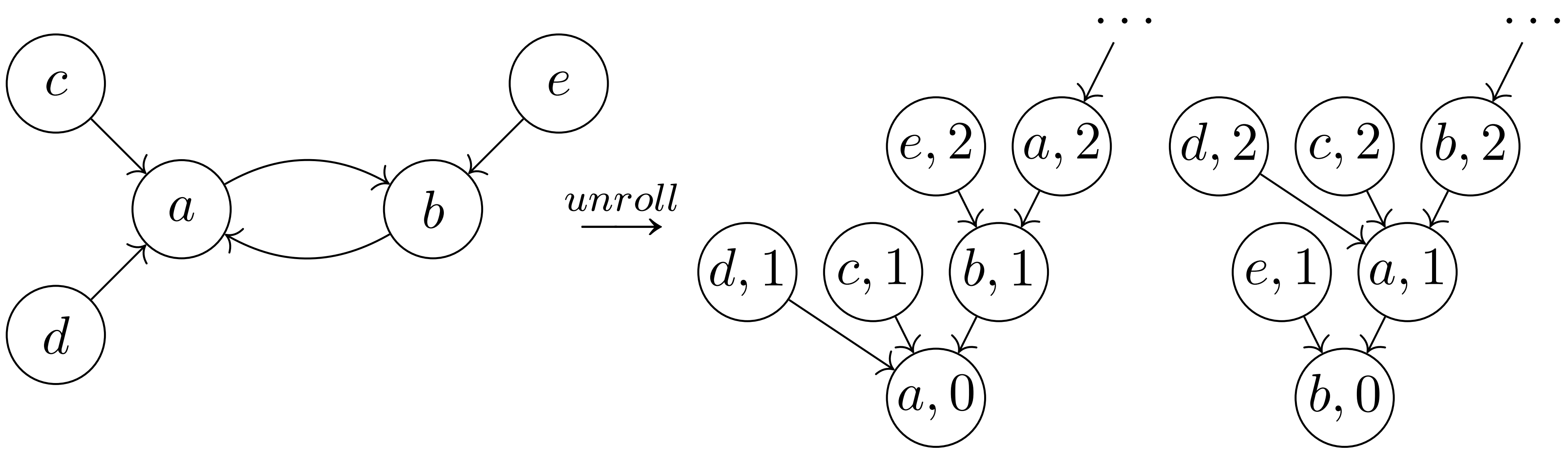
Problèmes prouvés faciles : dans les sommes de cycles

- Proposition Soit A et B deux sommes de cycles. Si A est fixé, alors on peut compter le nombre de solutions de $AX = B$ en temps polynomial.

- Proposition Soit A une somme de cycles et B une somme de plusieurs fois le même cycle. Résoudre $AX = B$ est polynomial si A et B sont donnés comme des graphes. Dans certains cas, polynomial si la taille des cycles est donnée en binaire.

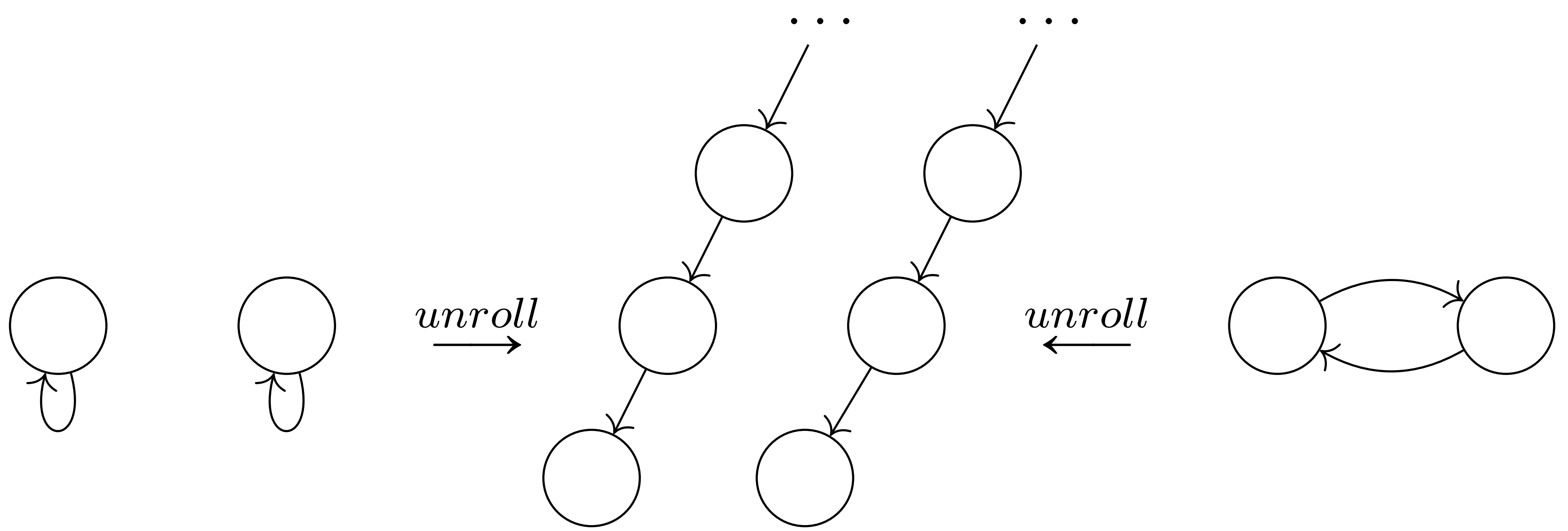
Au-delà des sommes de cycles

- Unroll : Idée Générale Ce sont des arbres qui correspondent à l'ensemble des pré-images possibles de chaque nœud des cycles.



Comportement des transients : unroll

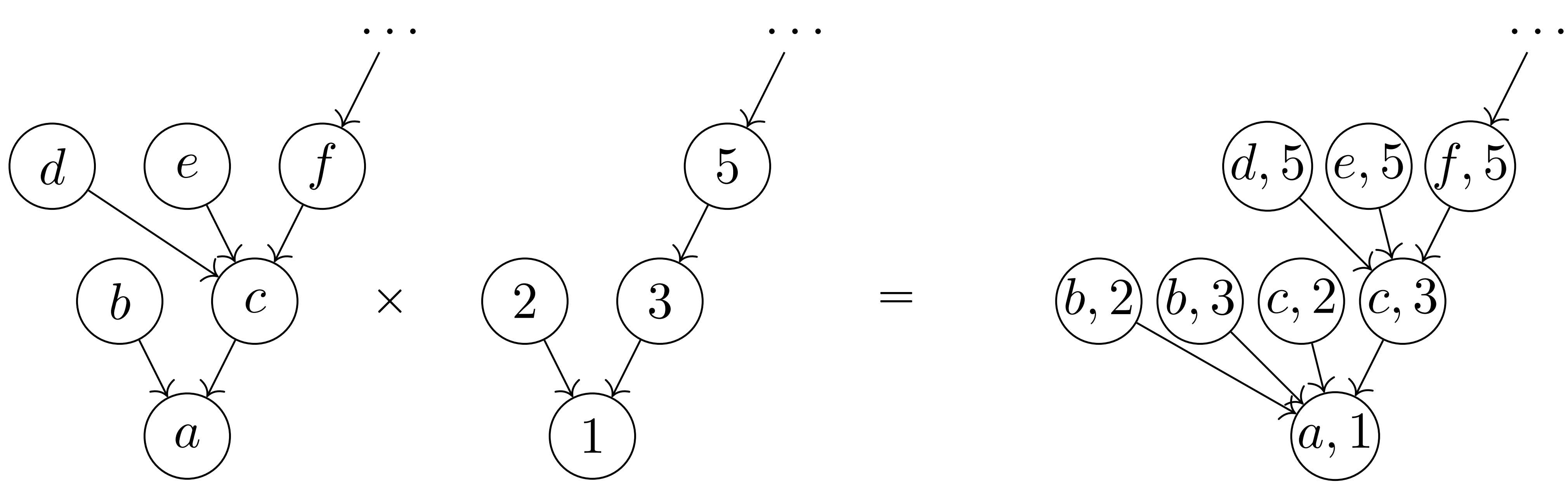
- Attention L'opération d'unroll n'est pas injective.



Opérations sur les unrolls

- Somme : Idée Générale Comme pour les systèmes dynamiques.

- Produit : Idée Générale Comme pour les systèmes dynamiques mais profondeur par profondeur.



Principaux résultats sur les unrolls

- Proposition Tous les polynômes d'unrolls sont injectifs.

- Théorème On peut résoudre les équations polynomiales sur les unrolls

$$\sum_{i=0}^{m} \mathcal{U}(A_i)\mathcal{U}(X^i) = \mathcal{U}(B)$$

en temps polynomial.

Problèmes prouvés faciles : dans les systèmes généraux

- Proposition On peut trouver la solution connexe de $P(X) = B$ (si elle existe) en temps polynomial.

- Théorème Si P est un polynôme d'une variable injective, alors on peut résoudre $P(X) = B$ en temps polynomial.

Encore au-delà

Dans le cas des sommes de cycles, il existe une procédure pour résoudre les équations de la forme

$$\sum_{i=1}^{m} A_i X^i = B$$

en la divisant en équations de la forme $AX = C$

Merci de votre attention !